Hypercubic designs and applications

A. Thannippara¹, B. Kurian¹,
D.K. Ghosh², S.C. Bagui³, S. Mandal⁴

¹ Department of Statistics, St. Thomas College, Pala,
Kerala, India
² Department of Statistics, Saurashtra University, Rajkot,
Gujarat, India
³ Department of Mathematics and Statistics, University
of West Florida, Pensacola, FL, USA
(corresponding author, e-mail: sbagui@uwf.edu)
⁴ Department of Statistics, University of Manitoba,
Winnipeg, MB, Canada

Received: July 23, 2003; revised version: May 17, 2005

In this paper we develop relatively easy methods for constructing hypercubic
designs from symmetrical factorial experiments for \( t = v^m \) treatments with
\( v = 2, 3 \). The proposed methods are easy to use and are flexible in terms of
choice of possible block sizes.

Key Words: Hypercubic design (HCD), E-optimal RGD, Partially balanced
incomplete block design (PBIBD), Extended-group divisible partially balanced
incomplete block design (EGD-PBIBD), Semi-regular graph design (SRGD).

1. Introduction

Literature on partially balanced incomplete block designs with two or more
than two associate classes includes work by Raghavarao and Chandrasekhararao
(1964), Vartak (1955), Hinkelmann (1964), Kusumoto (1965), and Aggarwal
(1974). Most of these articles developed various aspects of PBIB designs. Shah
(1958) introduced a design called hypercubic design. An incomplete block design
is said to be a hypercubic design in which \( t = v^m \) treatments are represented by
\( m \)-plets \((x_1, x_2, \ldots, x_m)\), where \( 1 \leq x_i \leq v, i = 1, 2, \ldots, m \), and two treatments
are said to be \( i \)th associates if they differ in exactly \( i \) components, \( i = 1, 2, \ldots, m \).
One of the criteria for the non-existence of a hypercubic design depends on the
negativity of the eigenvalues of its concurrence matrix \( NN' \). Due to the fact
that EGD-PBIBD can be reduced to a hypercubic design if the levels of all the
factors are the same, the association classes that have the same number of unity
components should have the same eigenvalues. Chang (1989) constructed some
hypercubic designs from symmetrical factorial experiments. He showed that all
normalized contrasts, belonging to the same main effect or interaction, can be
estimated with the same variance. This is an important property of the hyper-
cubic design, and it holds due to the fact that hypercubic design is a balanced
factorial experiment (BFE). In this article we propose two simple methods of
construction of hypercubic designs from symmetrical factorial experiments with
\( m \) factors, each at \( v \) level.
2. Methods of Construction

In this section we describe two methods for construction of hypercubic designs. One is for \( v = 2 \) and the other one is for \( v = 3 \). The range of block sizes of the hypercubic designs is restricted to \( 2 \leq k \leq 8 \). We also illustrated these methods through examples.

Method I. In this method, we describe how to obtain hypercubic designs with \( v = 2 \). Below we describe the method with parameters \( t = 2^m \), \( k = 2^h \) for \( h = 1, 2, \ldots, m-1 \), \( r = \binom{m}{h} \), \( b = \frac{2^m}{k} \), \( \lambda_i = \binom{m-i}{h-i} \) for \( i \leq h \) and \( \lambda_j = 0 \) for \( j > h \).

Suppose that all the \( 2^m \) treatment combinations are written in the standard sequence. For example, if the treatment combinations of \( 2^3 \) factorial experiment are written as \( 111, 112, 121, 122, 211, 212, 221, 222 \), we call it a standard sequence. Now the \( 2^m \) treatment combinations are assigned by \( t_1, t_2, \ldots, t_{2^m} \), in the corresponding standard sequence. The blocks of different replications are formed in the following manner:

Let \( S = \{2^0, 2^1, 2^2, \ldots, 2^{m-1}\} \) where the total number of elements in the set \( S \) is \( m \). Choose any \( h \) elements from the set \( S \) and denote this set by \( A = \{2^{j_0}, 2^{j_1}, 2^{j_2}, \ldots, 2^{j_{h-1}}\} \), where \( j_0, j_1, j_2, \ldots, j_{h-1} \) are any \( h \) distinct integers from \( 0, 1, 2, \ldots, m-1 \). Complement of \( A \) will have the other (remaining) indices from \( 0, 1, 2, \ldots, m-1 \), say, \( J_h, J_{h+1}, J_{h+2}, \ldots, J_{m-1} \) for \( 2 \). The treatments of any block in the corresponding replication are \( t_i, t_{i+2^0}, t_{i+2^1}, t_{i+2^2}, t_{i+2^0+2^1}, t_{i+2^0+2^1+2^2}, \ldots \) i.e., from the set \( A \), consider all possible combinations of \( h \) elements by taking 0 at a time, 1 at a time, 2 at a time, and so on up to \( h \) at a time. Then, find the sum of the elements in each combination, and add \( i \) to these sums. One gets a block of \( 2^h \) plots by taking treatment combinations having suffixes equal to these sums. The number of different combinations of \( h \) elements is \( \binom{h}{0} + \binom{h}{1} + \cdots + \binom{h}{h} = 2^h \). Hence, there are \( 2^h \) elements in a block, thus \( k = 2^h \). A proper choice of the values of \( i \) will provide all the blocks in a replication. The choice of \( i \) are as follows:

Let the complement of the set \( A \) be \( A^c = \{2^{J_h}, 2^{J_{h+1}}, 2^{J_{h+2}}, \ldots, 2^{J_{m-1}}\} \). Note that there are \( (m-h) \) elements in the set \( A^c \). As before, from set \( A^c \), consider all possible combinations of \( (m-h) \) elements by taking 0 at a time, 1 at a time, 2 at a time, and so on, up to \( (m-h) \) at a time. Find the sum of elements in each combination, and add 1 to these sums, we get the values of \( i \) as \( 1, 1+2^h, 1+2^h+2^2, \ldots \). Thus there are \( 2^{m-h} \) blocks in a replication. From \( m \) elements of the set \( S \), \( h \) elements can be selected in \( \binom{m}{h} \) ways, hence \( r = \binom{m}{h} \). It is obvious that \( b = \frac{2^m}{k} \), hence \( b = \binom{m}{h} 2^{m-h} \). Because of binary representations of the treatment labels, any pair of treatments in the same block can only be \( i \)-th associates if \( i \leq h \). In the construction, the blocks are formed using the set \( A \) where all possible combinations of \( h \) elements out of \( m \) elements are present. Furthermore any two treatments of HCD are of the \( i \)-th associate class if they are present in the same block and differ in exactly \( i \) components. Such pairs of \( i \)-th associate class treatments occur in \( \binom{m-i}{h-i} \) ways for \( i \leq h \). Hence \( \lambda_i = \binom{m-i}{h-i} \), for \( i \leq h \) and the remaining pairs of treatments will occur \( \lambda_j = 0 \) times for \( j > h \) as they are not present in the same block.