Regular A-optimal design matrices $X = (x_{ij})$ with $x_{ij} = -1, 0, 1$

Małgorzata Graczyk

Received: 19 May 2008 / Accepted: 15 June 2009 / Published online: 5 August 2009
© Springer-Verlag 2009

Abstract New construction methods of the regular A-optimal design matrices with elements $-1, 0, 1$ are presented, under assumption of nonhomogeneity of variance error. The presented constructions are based on the incidence matrices of the balanced bipartite weighing designs.

Keywords A-optimal design · Balanced bipartite weighing design

1 Introduction

Assume that the $n \times 1$ observation—vector $y$ follows a standard linear model

$$E(y) = Xw, \quad \text{Cov}(y) = \sigma^2 G,$$

where $X = (x_{ij})$ is an $n \times p$ matrix with known coefficients, usually called the design matrix, with $x_{ij} = -1, 0, 1, i = 1, 2, \ldots, n, \quad j = 1, 2, \ldots, p$, $w$ is a $p \times 1$ vector representing unknown parameters, $\sigma^2$ is a constant variance error, $G$ is an $n \times n$ positive definite diagonal matrix of known elements.

In the literature, the design matrix $X$ with elements $-1, 0, 1$ is called weighing matrix, and the relevant design is called a chemical balance weighing design, see Wong and Masaro (1984). Some optimality criterions and existence conditions determining such designs are given in Ceranka and Graczyk (2003), Ceranka and Katulska (2001), and Raghavarao (1971).
If $X'G^{-1}X$ is nonsingular then all parameters are estimable and

$$\hat{w} = \left( X'G^{-1}X \right)^{-1} X'G^{-1}y, \quad \text{Var}(\hat{w}) = \sigma^2 \left( X'G^{-1}X \right)^{-1}.$$ 

Various problems concerning the optimal designs are presented in Pukelsheim (1993) and Shah and Sinha (1989).

2 A lower bound for $\text{tr}(X'G^{-1}X)^{-1}$

For $X$ and $G$ we get from Ceranka and Katulska (2001) that

$$\text{tr}\left( X'G^{-1}X \right)^{-1} \geq \frac{p}{\text{tr}(G)},$$

where the equality holds if and only if $X'G^{-1}X$ is a scalar multiple of $I_p$, where $I_p$ denotes the $p \times p$ identity matrix.

**Definition 2.1** The design $X$ with the matrix $G$ is called a regular A-optimal design if the sum of variances of estimators, i.e. $\text{tr}(X'G^{-1}X)^{-1}$, attains the lower bound given in (1).

Also from Ceranka and Katulska (2001) we have

**Theorem 2.1** A design $X$ with the matrix $G$ is a regular A-optimal design if and only if

$$X'G^{-1}X = \text{tr}(G^{-1})I_p.$$ 

Now, let us consider some special forms of the matrix $G$, because the optimality conditions depend on this matrix. In the present paper we consider regular A-optimal design $X$ is as follows

$$X = \begin{bmatrix} I_{n_1} & X_1 \\ I_{n_2} & X_2 \end{bmatrix}$$

with the matrix $G$ of the form

$$G = \begin{bmatrix} aI_{n_1} & 0_{n_1} & 0_{n_2}' \\ 0_{n_2} & 0_{n_1} & I_{n_2} \end{bmatrix}, \quad a > 0, \quad a \neq 1, \quad n = n_1 + n_2,$$

where $1_s$ denotes the $s \times 1$ vector of ones.

For $X$ in (3) and $G$ in (4) we obtain from Ceranka and Katulska (2001)

**Theorem 2.2** A design $X$ given by (3) with the matrix $G$ in (4) is a regular A-optimal design if and only if