Characterizations through reliability measures from weighted distributions

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Received: December 7, 1998; revised version: May 10, 2000

Abstract

In this paper we obtain general characterizations of probability distributions from relationships between failure rate and mean residual life from the original distribution and associated weighted distribution. Our characterization properties extend particular results given by Gupta and Keating (1986), Jain et al. (1989) and Asadi (1998). Using the theoretical results we obtain characterizations of some usual distributions.

Key words: Characterization, Failure Rate, Mean Residual Life, Weighted and Length Biased distributions.

AMS subject classification: 62G05.

1 Introduction

In the last fifteen years, several interesting results concerning characterization problems have been published. The first results have been based on the failure rate function \( r(t) = \frac{f(t)}{1 - F(t)} \), mean residual life \( e(t) = E(X - t \mid X \geq t) \) and generalized left truncated mean function \( m_h(t) = E(h(X) \mid X \geq t) \) (see Vartak (1974), Kotz and Shanbhag (1980) and Zoroa et al. (1990)). When \( h(x) = x \), then \( m(t) = E(X \mid X \geq t) \) is known as the vitality function or left truncated mean function. It is well known that \( r(x) \), \( e(x) \) and \( m_h(x) \) uniquely determine the distribution function \( F(x) \). For example, for \( m_h(t) \) the inversion formula is

\[
F(x) = 1 - \exp\left( - \int_{-\infty}^{x} \frac{d m_h(t)}{m_h(t) - h(t)} \right)
\]

(see Zoroa et al. (1990)). Hence these three functions are equivalent, in the sense that given one, the other two can be determined. Recently, new characterizations of probabilistic models have been given from lineal relations between

\*Supported by D.G.E.S. under Grant PB96/1105
these three function (see Kotz and Shanbhag (1980), Nair and Sankaran (1991) or Ruiz and Navarro (1994)).

Associated to a r.v. $X$ with survival (reliability) function $R_X(t) = \Pr(X \geq t)$ and to a positive real function $w(t)$, it defines the weighted r.v. $Y$ with density function

$$f_Y(t) = \frac{w(t)f_X(t)}{E(w(X))} \tag{2}$$

where $f_X(t)$ is the density function of $X$ and $E(w(X)) < \infty$. The concept of weighted distribution was formulated by Rao (1965) to model various situations in which the sampling probabilities are proportional to a "weighted" function $w$. In particular, if $w(t) = t$, then $Y$ is the length (or size) biased r.v. which represents size biased samples. The usefulness of weighted and length biased r.v. can be seen in Patil and Rao (1977) and Gupta and Kirmani (1990).

Recently, Gupta and Keating (1986), Jain et al. (1989) and Asadi (1998) have studied the characterization of some particular distributions through relationships between reliability measures of weighted r.v. $Y$ and parent r.v. $X$. In this paper, we obtain a general way to obtain the distribution function $F_X$ from relationships such as those considered by Gupta and Keating (1986), Jain et al. (1989) and Asadi (1998). From our results, the particular characterizations given by these authors can be obtained.

In particular, in section 2, we obtain the distribution function $F_X$ from the ratio of survival functions ($R_Y/R_X$), failure rates ($r_Y/r_X$) or the product of generalized left truncated mean functions ($m_{w,Y} \cdot m_{w,X}$). Moreover, we obtain a particular characterization result for the equilibrium distribution in a renewal process. In section 3, we obtain the density function $f_X$ from relationships between failures rates and vitality functions of $X$ and $Y$, thus extending the results given in Asadi (1998). Finally, in section 3 we apply our results to obtain the characterization of some particular distributions.

2 Characterization theorems from linear relations.

Henceforth, let $X$ be an absolutely continuous r.v., let $w(t)$ be a positive, non constant and continuous real function with $E(w(X)) < \infty$ and let $Y$ be the weighted r.v. associated with $X$ and $w(t)$, whose density function is defined by (2). Note that if $w(x)$ is constant, then $X \equiv Y$ (trivial case).

**Theorem 1** If $\alpha(t) = R_Y(t)/R_X(t)$ and $k = E(w(X))$, then

$$R_X(x) = \exp \left\{ - \int_{-\infty}^{x} \frac{\alpha'(t)dt}{\alpha(t) - w(t)/k} \right\} \tag{3}$$