Approximation in $L_p(\mathbb{R}^d)$ from a Space Spanned by the Scattered Shifts of a Radial Basis Function

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Abstract. A new multivariate approximation scheme on $\mathbb{R}^d$ using scattered translates of the "shifted" surface spline function is developed. The scheme is shown to provide spectral $L_p$-approximation orders with $1 \leq p \leq \infty$, i.e., approximation orders that depend on the smoothness of the approximands. In addition, it applies to noisy data as well as noiseless data. A numerical example is presented with a comparison between the new scheme and the surface spline interpolation method.

1. Introduction

1.1. General

Approximation schemes of the form

$$s(x) := \sum_{\xi \in \Xi} c_\xi \varphi(x - \xi), \quad x \in \mathbb{R}^d,$$

with $\varphi$ a "suitable" basis function are known to be effective for approximation to scattered data. The use of a radially symmetric basis function $\varphi$ is particularly useful because this facilitates the evaluation of the approximant, while still leaving enough flexibility in the choice of $\varphi$. The set $\Xi$ in $\mathbb{R}^d$ by which the radial basis function $\varphi$ is shifted is usually referred to as a set of "centers." The common choices of $\varphi$ include: $\varphi(x) = |x|^d \log |x|$ with $d$ and $\lambda$ both even integers (surface spline), $\varphi(x) = (|x|^2 + c^2)^{\lambda/2}$ with $\lambda$ and $d$ both odd integers (multiquadric), and $\varphi(x) = \exp(-c|x|^2)$, $c > 0$ (Gaussian).

The initial approximation method using radial basis functions has been obtained by means of interpolation at finitely many scattered points $\Xi$ in $\mathbb{R}^d$. However, while the interpolation method is certainly an important approach toward solving the scattered data problem, it has several drawbacks. For example, for a large class of basis functions (including multiquadrics and inverse multiquadrics), the existing theories guarantee the interpolant to approximate well for only a very small class of very smooth approximands (see [MN2]). Another drawback of the interpolation method is connected with the issue of numerical stability: as the number of centers increases, one needs to solve a large linear...
system which is ill-conditioned. Last but not least, interpolation is never recommended when the data are known to be contaminated (= noisy data). All in all, there is an overwhelming need for approximation methods other than interpolation.

In view of the above discussion, we need a scheme with the following properties:

(i) it should approximate well a large class of functions;
(ii) it should be "local," namely, a coefficient in (1.1) should be determined by a few values of the data, even when many centers are involved in the scheme; and
(iii) the scheme should have a "smoothing" effect.

Thus, the main objective of this paper is, indeed, to construct an approximation scheme on nonuniformly distributed centers that satisfies all the above (and more).

It should be noted that noninterpolatory approximation schemes of the type (1.1) are also discussed intensively in the literature. However, most of the results in that direction deal with the case when the center set Ξ is infinite and uniform, i.e., a scale δZd of the integer lattice Zd. In fact, there are only a handful of treatments of noninterpolatory schemes for arbitrary center sets Ξ. Buhmann, Dyn, and Levin [BuDL] were among the first to construct a noninterpolatory approximation scheme for infinitely many scattered centers and to analyze its approximation power. Dyn and Ron [DR] showed that the scheme in [BuDL] can be understood as "an approximation to a uniform mesh approximation scheme." In both papers, quasi-interpolation schemes from radial basis function spaces with infinitely many centers Ξ were studied and both showed that the approximation orders obtained in the scattered case are identical to those that had been known on uniform grids. In particular, N. Dyn and A. Ron provided a general tool that allows us to convert any known approximation scheme on uniform grids to nonuniform grids, while preserving (to the extent that this is possible) the approximation orders known in the former case. The approach of [DR] can be described as follows: suppose that we are given an approximation scheme

\[ f \mapsto \sum_{\alpha \in \mathbb{Z}^d} \lambda_\alpha(f) \varphi(\cdot - \alpha). \]

Then, we replace each \( \varphi(\cdot - \alpha) \) by a suitable combination

\[ \sum_{\xi \in \Xi} A(\alpha, \xi) \varphi(\cdot - \xi), \]

with \( \Xi \) the set of scattered centers we wish to use. This result, however, requires one to choose the density of a uniform grid that is associated with the given scattered set \( \Xi \). As an alternative, we construct in this paper a new approximation scheme that, while based on the general idea of [DR], is not connected to uniform grid approximations. The approximation scheme that is developed and analyzed here is intrinsically "scattered": it employs directly the scattered shifts of the basis function \( \varphi \). Furthermore, while the conversion tool in [DR] is applied there only to stationary schemes (see [BR], [DJLR]), we successfully apply our new scheme to the more general nonstationary case. This results in schemes that provide spectral approximation orders (i.e., approximation orders that depend only on the smoothness of the approximands \( f \) we approximate).

Before we advance our discussion further, we would like to comment on the notion of "radial basis function," a comment which, as a matter of fact, is valid for all studies in