1 Introduction

In recent years, the visualization of mathematical concepts and ideas has attracted the attention of a growing number of researchers [6, 14, 15, 19, 28, 29, 36, 43, 44]. Mathematical visualization refers to the science and art of creating computer-based graphical representations of abstract mathematical objects and concepts to assist in their understanding [19, 29]. Mathematical visualization is especially effective for areas of mathematics in which the information content is non-trivial and insight can be gained by discovering visual patterns and relationships, beyond what can be studied algebraically [19].

Mathematical visualization tools should allow users to create images of unknown domains for which it is difficult or impossible to create physical models. This visualization process is used to help users construct new mental models of the underlying mathematical ideas and objects. Although mathematical visualization can serve many purposes, including mathematical and scientific enquiry and discovery, some suggest that its principal contribution will be pedagogical [5, 28].

With a few exceptions, in the design of computational visualization tools the emphasis has traditionally been on encoding and rendering the information without much creative input from the user [e.g., 13, 14, 15, 19, 31, 35, 43]. Effectiveness of any visualized image depends, to a large degree, on how its users interpret it and make sense of its inherent semantic meaning [4, 22, 37, 38]. Making sense of mathematical visualizations often depends on a dialectic reasoning process in which the user observes the visual image, poses “what if” questions to it, and experiments with it until its meaning is gradually understood [40]. This process can be mediated and supported through appropriate user interfaces.

User interfaces play a crucial role in how the user can process the visualized information [38, 39]. Through the user interface, the user can interact with the visual image: alter it, deform it, manipulate it, experiment with it, receive feedback from it, pose “what if” queries to it, and rearrange its information [1, 27, 28, 37, 42]. Such interaction can enormously facilitate the acquisition of qualitative and quantitative insight into the meaning and patterns of the visualized image. Research suggests and demonstrates that adding responsive interaction to static visual representations facilitates active exploration of the visuals and enhances their communicative effectiveness [12, 34, 37, 38, 42]. In interactive visualization tools, an image is not rendered once to be looked at; it is...
constructed and reconstructed in an iterative manner until its underlying relationships and structures are perceived [37]. This process of transforming images iteratively allows users to understand their inherent meaning in an evolving manner [38].

The process of mathematics exploration – mathematics being the science of patterns and order and the systematization of relationships [32] – can greatly benefit from the addition of interactivity [44]. Using interactive mathematical visualization tools, users feel that they are “doing” mathematics rather than “watching” mathematics [44]. This is especially true of images that have great interiority of meaning, images that require reflective, mental elaboration to penetrate the depth of their meaning, and images that have structural complexity and layers of abstraction [27, 34, 38]. Interaction can support the process of mental elaboration and manipulation of visualized objects by offloading some of the computational cost of thinking onto the computer.

Four-dimensional geometry is an area of higher mathematics that particularly requires visual representation of objects for a better understanding of the mathematical content. The complex structure of these objects makes their visualization and mental manipulation an intriguing and challenging intellectual task [18]. Many techniques have been developed and proposed for visualization of higher dimensional geometry. Most of these techniques deal with how to visually render this type of information, such as shading, projecting, plane-tracing, patching, wireframe rendering, ceramic rendering, surface thickening, and lighting, among other techniques [16–18, 29]. However, devising interactive techniques for dynamic exploration of 4D geometry has not received much attention, despite the fact that dynamic geometry tools for low-dimensional geometry have proven to dramatically affect how geometric concepts are understood [23, 36]. There exist some techniques to support interaction with high-dimensional geometry, such as clipping, contouring, ribboning, sidewalk walking, slicing, and n-dimensions ball rolling [3, 19–21]. However, most of these techniques focus on continuous mathematical spaces, such as 3-manifolds and knotted surfaces. Since their inception more than 2000 years ago, there has been a recent revival of research and interest in polytopes [25, 46]. The study of polytopes lies in the intersection of combinatorics, group theory (study of symmetries), and discrete geometry. Four-dimensional polytopes are discrete structures like the lower dimensional polytopes, polygons and polyhedra. A hypercube is a simple example of a 4D polytope (Fig. 1a).

Four-dimensional polytopes are comprised of lower-dimensional discrete components: vertices, edges, faces, and cells. These components define the surface of the polytope. The vertices are critical points of the polytope that define the surface. The edges, faces, and cells are the lines, polygons, and polyhedra, respectively, that fill in the surface by joining the vertices. A hypercube, for instance, is comprised of 16 vertices, 32 edges, 24 square faces, and eight cubic cells. As the number of components of a polytope increases, its 2D representation grows in visual complexity and becomes prohibitively difficult to mentally visualize, manipulate, and interpret. For instance, an omnitruncated hypercube, which is the most complex polytope, has 384 vertices, 768 edges, 464 faces, and 80 cells (Fig. 1b).

In this paper we present an overall approach for the dynamic exploration of uniform 4D polytopes. This approach encompasses a number of interactive techniques that allow for the dynamic adjustment of the degree of visual complexity of 4D polytope images. This approach is presented in the context of a software application called Polyvise – Polytope Visualization Explorer. The paper will first cover the mathematical background required to understand the context in which the approach is explored. Then there will be a brief overview of the tool followed by an in-depth discussion of the techniques employed that are derived from our approach. Finally, some scenarios will be presented suggesting the sophisticated level of interaction with the mathematical images afforded by the tool.

2 Mathematical background

Since their inception more than 2000 years ago, there has been a recent revival of research and interest in polytopes [25, 46]. The study of polytopes lies in the intersection of combinatorics, group theory (study of symmetries), and discrete geometry. Four-dimensional polytopes are discrete structures like the lower dimensional polytopes, polygons and polyhedra. A hypercube is a simple example of a 4D polytope (Fig. 1a).

Four-dimensional polytopes are comprised of lower-dimensional discrete components: vertices, edges, faces, and cells. These components define the surface of the polytope. The vertices are critical points of the polytope that define the surface. The edges, faces, and cells are the lines, polygons, and polyhedra, respectively, that fill in the surface by joining the vertices. A hypercube, for instance, is comprised of 16 vertices, 32 edges, 24 square faces, and eight cubic cells. As the number of components of a polytope increases, its 2D representation grows in visual complexity and becomes prohibitively difficult to mentally visualize, manipulate, and interpret. For instance, an omnitruncated hypercube, which is not the most visually complex polytope, has 384 vertices, 768 edges, 464 faces, and 80 cells (Fig. 1b).

The above polytopes (Fig. 1) have a high level of structural symmetry. This symmetry is due to the uniform nature of these polytopes, that is, all their