On the Crossing Number of $K_{m} \square P_{n}$

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Abstract. Crossing numbers of graphs are in general very difficult to compute. There are several known exact results on the crossing number of the Cartesian products of paths, cycles or stars with small graphs. In this paper we study $cr(K_{m} \square P_{n})$, the crossing number of the Cartesian product $K_{m} \square P_{n}$. We prove that $cr(K_{m} \square P_{n}) \leq \frac{1}{2} \left( \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m+1}{2} \right\rfloor (m - 1) + \left\lfloor \frac{m+2}{2} \right\rfloor \right)$ for $m \geq 3$, $n \geq 1$ and $cr(K_{m} \square P_{n}) \geq (n - 1)cr(K_{m+2} - e) + 2cr(K_{m+1})$. For $m \leq 5$, according to Křeček, Jendroľ and Ščerbová, the equality holds. In this paper, we also prove that the equality holds for $m = 6$, i.e., $cr(K_{6} \square P_{n}) = 15n + 3$.

Key words. Crossing number, Cartesian product, Complete graph, Path

1. Introduction

We consider only finite undirected graphs without loops or multiple edges.

Let $G$ be a graph with vertex set $V$ and edge set $E$. A drawing is a mapping of a graph into a surface. Vertices go into distinct nodes. An edge and its incident vertices map into a homeomorphic image of the closed interval $[0,1]$ with the relevant nodes as endpoints and the interior, an arc, containing no node. We consider only good drawings of a graph, i.e., a drawing satisfying the following conditions: (1) no edge crosses itself, (2) adjacent edges do not cross, (3) crossing edges do so only once, (4) edges do not cross vertices and (5) no more than two edges cross at a common point. A common point of two arcs is a crossing. An optimal drawing in a given surface is a good drawing which attains the smallest number of crossings. This number is the crossing number of the graph for the surface. We denote the crossing number of $G$ for the plane (equivalently, cylinder or sphere) by $cr(G)$. If $D := D(G)$ is a good drawing of $G$, then $v(D)$ denotes the number of the crossings in $D(G)$. We also speak about the nodes as vertices and the arcs as edges. It is clear that $cr(G) \leq v(D)$. A local rotation $\rho(v)$ of vertex $v$ is a (counterclockwise) cyclic permutation of the edge ends at $v$ in $D$. A rotation of $D$ is a collection of local rotations. The Cartesian product $G \square H$ of graphs $G$ and $H$ has vertex set $V(G) \times V(H)$ and edge set $E(G \square H) = \{((x_1, y_1), (x_2, y_2)) \mid x_1 = x_2 \text{ and } y_1 y_2 \in E(H) \text{ or } y_1 = y_2 \text{ and } x_1 x_2 \in E(G)\}$. (In the references of this paper, the authors use $G \times H$ to represent the Cartesian product of graphs $G$ and $H$.)

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It is well known that \( cr(K_m) \leq \frac{1}{4} \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-2}{2} \right\rfloor \left\lfloor \frac{m-3}{2} \right\rfloor \). The equality has been established only for \( n \leq 12 \) [2]. Harary et al. conjectured that \( cr(C_m \boxtimes C_n) = (m-2)n \), for \( 3 \leq m \leq n \). This has been verified for \( m \leq 7 \) [1, 12–15]. Glebsky and Salazar [16] also showed that the conjecture holds for \( n \geq m(m+1) \) and \( m \geq 3 \).

Klešč [8] determined the crossing numbers of products of all 4-vertex graphs with paths and stars except \( cr(K_{1,3} \Box P_6) \), which is determined by Jendrol and Ščerbová [7]. Beineke and Ringel [4, 5] determined the crossing numbers of products of all 4-vertex graphs with cycles.

Klešč [10, 11] has determined the crossing numbers of products of all 5-vertex graphs with paths. In particular he proved that \( cr(K_5 \boxtimes P_n) = 6n \) in [9].

In this paper we study the crossing numbers of \( K_m \Box P_n \). We prove that \( cr(K_m \Box P_n) \leq \frac{1}{4} \left\lfloor \frac{m+1}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m-2}{2} \right\rfloor \left\lfloor \frac{m-3}{2} \right\rfloor (n \left\lfloor \frac{m+4}{2} \right\rfloor + \left\lfloor \frac{m-4}{2} \right\rfloor) \) for \( m \geq 3, n \geq 1 \) and \( cr(K_m \Box P_n) \geq (n-1)cr(K_{m+2} - e) + 2cr(K_{m+1}) \). For \( m \leq 5 \), according to Klešč, Jendrol and Ščerbová, the equality holds. In this paper, we also prove that the equality holds for \( m = 6 \), i.e., \( cr(K_6 \Box P_n) = 15n + 3 \).

2. Upper Bounds of \( cr(K_m \Box P_n) \)

Let
\[
V(K_m \Box P_n) = \{ v_j^i \mid 0 \leq j \leq m - 1, 0 \leq i \leq n \},
\]
\[
E(K_m \Box P_n) = \left( \bigcup_{i=0}^{n} \{ v_j^i v_k^i \mid 0 \leq j < k \leq m - 1 \} \right) \cup \left( \bigcup_{i=1}^{n} \{ v_j^{i-1} v_j^i \mid 0 \leq j \leq m - 1 \} \right).
\]
Let \( V^i = \{ v_j^i \mid 0 \leq j \leq m - 1 \} \), \( E^i = \{ v_j^i v_k^i \mid 0 \leq j < k \leq m - 1 \} \), \( K_m^i = (V^i, E^i) \) for \( 0 \leq i \leq n \). Let \( P^i = \{ v_j^{i-1} v_j^i \mid 0 \leq j \leq m - 1 \} \) for \( 1 \leq i \leq n \). Then, we have
\[
E^i \cap E^j = \emptyset, \quad 0 \leq i < j \leq n,
\]
\[
P^i \cap P^j = \emptyset, \quad 1 \leq i < j \leq n,
\]
\[
E^i \cap P^j = \emptyset, \quad 0 \leq i \leq n, 1 \leq j \leq n,
\]
\[
E(K_m \Box P_n) = \left( \bigcup_{i=0}^{n} E^i \right) \cup \left( \bigcup_{i=1}^{n} P^i \right).
\]

Let \( A, B \) be two disjoint subsets of \( E(G) \). In a drawing \( D \), the number of the crossings that involve an edge in \( A \) and another edge in \( B \) is denoted by \( v_D(A, B) \).

The number of the crossings that involve a pair of edges in \( A \) is denoted by \( v_D(A) \). So \( v(D) = v_D(E(G)) \). If an edge is not crossed by any other edge, we say that it is clean in \( D \); if it is crossed by at least one edge, we say that it is crossed in \( D \).

Let \( X \) be a subset of \( V(G) \) or of \( E(G) \) for a graph \( G \). Then \( G[X] \) denotes the subgraph of \( G \) induced by \( X \). The following two statements are straightforward.

**Lemma 2.1.** Let \( A, B, C \) be mutually disjoint subsets of \( E(G) \). Then,
\[
v_D(C, A \cup B) = v_D(C, A) + v_D(C, B),
\]
\[
v_D(A \cup B) = v_D(A) + v_D(B) + v_D(A, B).
\]