On the Convexity Number of Graphs

Mitre C. Dourado · Fábio Protti · Dieter Rautenbach · Jayme L. Szwarcfiter

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Abstract A set of vertices $S$ in a graph is convex if it contains all vertices which belong to shortest paths between vertices in $S$. The convexity number $c(G)$ of a graph $G$ is the maximum cardinality of a convex set of vertices which does not contain all vertices of $G$. We prove NP-completeness of the problem to decide for a given bipartite graph $G$ and an integer $k$ whether $c(G) \geq k$. Furthermore, we identify natural necessary extension properties of graphs of small convexity number and study the interplay between these properties and upper bounds on the convexity number.

Keywords Convexity number · Convex hull · Convex set · Graph · Shortest path

M. C. Dourado (✉)
Instituto de Matemática, Universidade Federal do Rio de Janeiro,
Rio de Janeiro, RJ, Brazil
e-mail: mitre@dcc.ufrj.br

F. Protti
Instituto de Computação, Universidade Federal Fluminense,
Niterói, RJ, Brazil
e-mail: fabio@ic.uff.br

D. Rautenbach
Institut für Optimierung und Operations Research, Universität Ulm,
89069 Ulm, Germany
e-mail: dieter.rautenbach@uni-ulm.de

J. L. Szwarcfiter
Instituto de Matemática, NCE and COPPE, Universidade Federal do Rio de Janeiro,
Rio de Janeiro, RJ, Brazil
e-mail: jayme@nce.ufrj.br
1 Introduction

We consider finite, undirected and simple graphs $G$ with vertex set $V(G)$ and edge set $E(G)$. For two vertices $u$ and $v$ of a graph $G$, let $I[u, v]$ denote the set of vertices of $G$ which belong to a shortest path between $u$ and $v$ in $G$. For a set of vertices $S$, let $I[S]$ denote the union of the sets $I[u, v]$ over all pairs of vertices $u$ and $v$ in $S$. A set of vertices $S$ is convex if $I[S] = S$. The convex hull $H[S]$ of a set $S$ of vertices is the smallest convex set of vertices which contains $S$. Since the intersection of two convex sets is convex, the convex hull is well defined.

Chartrand, Wall, and Zhang [4] define the convexity number $c(G)$ of a graph $G$ as the largest cardinality of a convex set of vertices which does not contain all vertices of $G$. Gimbel [8] proved that the decision problem associated to the convexity number is NP-complete. For further related results, we refer the reader to [2,3,5,6,9,10].

Our contributions in the present paper concern the algorithmic complexity of the convexity number and the structure of graphs of small convexity number. In Sect. 2, we refine Gimbel’s hardness result [8] by proving NP-completeness for the class of bipartite graphs. Furthermore, we describe how to efficiently decide whether the convexity number is at least $k$ for some fixed $k$ and how to determine the convexity number for cographs in linear time. In Sect. 3, we study graphs of small convexity number. We identify natural necessary extension properties of such graphs and prove best possible upper bounds on the convexity number implied by these necessary conditions.

2 NP-Completeness for Bipartite Graphs

Our main result in this section is the NP-completeness of the following decision problem restricted to bipartite graphs.

**CONVEXITY NUMBER**

**Instance:** A graph $G$ and an integer $k$.

**Question:** Is $c(G) \geq k$?

We start by showing how to solve the above problem in polynomial time, for fixed $k$. Let $G$ be a graph and let $S$ be a set of vertices of $G$. By definition, $S$ is not convex if and only if there are two vertices $x$ and $y$ in $S$ such that $I[x, y] \not\subseteq S$. Choosing such a pair of vertices at minimum distance, we obtain that $S$ is not convex if and only if there are two vertices $x$ and $y$ in $S$ such that there exists a shortest path $P$ between $x$ and $y$ which is of length at least 2 and whose internal vertices all belong to $V(G) \setminus S$. Applying a shortest path algorithm to the induced subgraphs $G - (S \setminus \{x, y\}) = G[\{x, y\} \cup (V(G) \setminus S)]$ of $G$ for all pairs of distinct vertices $x$ and $y$ in $S$, such paths can be found in polynomial time. Furthermore, iteratively extending a non-convex set by the internal vertices of such paths, one can determine the convex hull of a set of vertices in polynomial time.

By definition, the convexity number of a graph $G$ is less than some integer $k$ if and only if the convex hull of every set of exactly $k$ vertices contains all vertices of $G$. Hence for fixed $k$, it can be decided in polynomial time whether the convexity number of a graph is at least $k$.

We proceed to our main result in this section.