Reconstructing a Graph from its Arc Incidence Graph

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Abstract    Introduced implicitly by Brualdi and Massey (Discret Math 122(1–3):51–58, 1993) in their work on the strong chromatic index of multigraphs, the arc incidence graph $AI(G)$ of a graph $G$ is defined as the square of the line graph of the incidence graph of $G$. We describe a linear-time algorithm for recognizing arc incidence graphs and reconstructing a graph with no isolated vertices from its arc incidence graph.

Keywords    Arc incidence graph · Incidence graph · Incidence coloring conjecture · Reconstruction Algorithm

1 Introduction

The arc incidence graph of a graph $G$, denoted $AI(G)$, was implicitly introduced by Brualdi and Massey [3] in their work on the strong chromatic index of multigraphs. In that context, Brualdi and Massey refer to the chromatic number of $AI(G)$ as the incidence coloring number of $G$, and because $AI(G)$ can be defined as the square of the line graph of the incidence graph of $G$, it follows that the incidence coloring number of $G$ equals the strong chromatic index of the incidence graph of $G$. Brualdi and Massey also formulated the so-called Incidence Coloring Conjecture, and as a
result, the incidence coloring numbers of various graphs (equivalently, the chromatic numbers of arc incidence graphs) have been studied in several papers (see, for example, Hosseini Dolama, Sopena, and Zhu [10] and Maydanskiy [13]). Venkataraman, Sundareswaran, and Swaminathan [17] studied the domination and independence numbers of arc incidence graphs, while Zhang et al. [18] studied (among other topics) their hamiltonicity.

In Sect. 2, we give a more concrete definition of $\text{AI}(G)$ which suggests that the arc incidence graph can be viewed as a variant of the line graph. Much work has been done on line graphs, including structural characterizations (see [2, 11, 16]), recognition algorithms, and hamiltonicity properties (see [4, 6]). In particular, Lehot [12] and Roussopoulos [14] present linear-time algorithms for reconstructing a graph from its line graph. In that spirit, this paper describes a linear-time algorithm for recognizing arc incidence graphs and reconstructing a graph with no isolated vertices from its arc incidence graph. The algorithm also shows that the map $G \mapsto \text{AI}(G)$ is injective for graphs $G$ with no isolated vertices. Section 2 contains preliminary material on arc incidence graphs, while Sect. 3 describes the algorithm and proves its correctness.

We close this section with the formal statement of our main result.

**Reconstruction Theorem** Let $H$ be a graph with $n$ vertices and $m$ edges. The Reconstruction Algorithm described in Sect. 3 either finds a graph $G$ with no isolated vertices such that $H = \text{AI}(G)$ or it determines that no such $G$ exists. If $G$ exists, it is unique up to isomorphism. The running time of the Reconstruction Algorithm is $O(m + n)$.

**Note 1.1** To be precise in the statement of the Reconstruction Theorem, we should specify that the running time relies on: (1) having a fixed, known (but arbitrary) linear ordering on the vertices in $H$; (2) being able to enumerate the vertices adjacent to a vertex $v \in H$ in $O(\deg_H v)$ steps in the linear order given by (1); and (3) being able to determine if two vertices in $H$ are adjacent in $O(1)$ steps. These three conditions are satisfied if we are given the adjacency lists of vertices in $H$ as linked hash sets in which the iteration orders are consistent with some fixed, known linear ordering of the vertices. Note that, in particular, this allows us to pre-compute the degrees of the vertices in $H$ as well as the maximum and minimum degrees in $H$ in $O(m)$ steps.

## 2 Preliminaries

It turns out to be more natural to define the arc incidence graph of a digraph before defining the arc incidence graph of a graph; in particular, this will explain the use of the word arc. Let $D$ be a digraph. Denote an arc in $D$ from $u$ to $v$ by $(u, v)$; we call $u$ its initial vertex and $v$ its terminal vertex. The arc incidence graph of $D$, denoted $\text{AI}(D)$, has the arcs of $D$ as its vertex set, and there is an edge between two vertices in $\text{AI}(D)$ if one of the following conditions holds: (1) the corresponding arcs in $D$ share the same initial vertex (we call this an edge of type 1); (2) the initial vertex of one arc is the terminal vertex of the other and vice versa (we call this an edge of type 2); or

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1 A linked hash set is a hash set that maintains a doubly linked list of its entries so that its iteration order is consistent (and is equal to insertion order). See [5, Exercise 22.1–8] and [1] for an implementation in Java.