The Existence of $FrGBT D(4, g^u)'s$

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Abstract If the blocks of a $GDD(X, G, A)$ with block size 4, index 3 and type $g^u$ can be arranged into a $(gu)/4 \times (gu)$ array, such that: (1) the main diagonal consists of $u$ empty subarrays of size $g/4 \times g$; (2) the blocks in each column form a partition of $X \setminus G$ for some $G \in G$, while the blocks in each row contains every element of $X \setminus G$ 3 times and no element of $G$ for some $G \in G$, then the design is called a frame generalized balanced tournament design and denoted by $FrGBT D(4, g^u)$. The necessary conditions for the existence of such a design are $u \geq 6$ and $g \equiv 0 (\text{mod} 4)$. In this paper, the sufficiency of these conditions is proved with some possible exceptions.

Keywords GBTDs · Frame GBTDs · GDDs

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1 Introduction

Let \( v, \lambda \) be positive integers, \( K \) be a set of positive integers. A *group divisible design*, denoted by \((K, \lambda)\)-GDD, is a triple \((X, \mathcal{G}, \mathcal{B})\), such that:

1. \( X \) is a \( v \)-set;
2. \( \mathcal{G} \) is a collection of nonempty subsets (called *groups*) of \( X \) which partition \( X \);
3. \( \mathcal{B} \) is a collection of subsets (called *blocks*) of \( X \) with \(|\mathcal{B}| \in K\) for any \( B \in \mathcal{B} \), such that every pair of points not contained in a group occurs in exactly \( \lambda \) blocks and no pair of points contained in a group occurs in any block.

The group type of a \((K, \lambda)\)-GDD is the multiset \( T = \{ |G| : G \in \mathcal{G} \} \) and we usually describe it by an exponential notation: type \( a^i b^j \ldots \) denotes that the design has \( i \) groups of size \( a \), \( j \) groups of size \( b \), and so on. When \( K = \{ k \} \), we simply write \( k \) for \( K \).

A \((k, 1)\)-GDD of type \( n^k \) is called a *transversal design* and denoted by \( TD(k, n) \). Denote by \( RTD(k, n) \) a \( 1\)-*resolvable transversal design*. The existence of a \( TD(k, n) \) is equivalent to the existence of an \( RTD(k - 1, n) \).

A \((K, \lambda)\)-GDD of type \( 1^v \) is called a *pairwise balanced design* and denoted by \( PBD(K, \lambda, v) \).

Let \( g, u, k \) be positive integers, and \( k \mid g, u \geq k + 2 \). For \( 0 \leq i, j \leq u - 1 \), define \( R_i = \{ w + (ig/k) : w = 0, 1, \ldots, (g/k) - 1 \} \), \( C_j = \{ s + jg : s = 0, 1, \ldots, g - 1 \} \). \((X, \mathcal{G}, \mathcal{A})\) is a \((k, k - 1)\)-GDD of type \( g^u \), where \( \mathcal{G} = \{ G_1, G_2, \ldots, G_u \} \). If the blocks of \( \mathcal{A} \) can be arranged in to a \(|X|/k \times |X|\) array \( F \) whose rows and columns are indexed by the elements of \( R_0, R_1, \ldots, R_{u - 1} \) and \( C_0, C_1, \ldots, C_{u - 1} \) in turn, such that:

1. Suppose that \( F_i(0 \leq i \leq u - 1) \) are the subarrays indexed by the elements of \( R_i \) and \( G_i \), then \( F_i(0 \leq i \leq u - 1) \) are all empty, that is to say that the main diagonal of \( F \) consists of \( u \) empty subarrays of order \( g/k \times g \);
2. For any \( r \in R_0(0 \leq i \leq u - 1) \), every point of \( X \setminus G_i \) occurs in exactly \( k \) blocks of row \( r \), while any point of \( G_i \) does not occurs in any block of row \( r \);
3. For any \( c \in C_j(0 \leq j \leq u - 1) \), every point of \( X \setminus G_j \) occurs in exactly one block of column \( c \), while any point of \( G_j \) does not occurs in any block of column \( c \), then the design is called a frame generalized balanced tournament design and denoted by \( FrGBT D(k, g^u) \).

The necessary conditions for the existence of such a design are \( u \geq k + 2 \) and \( k \mid g \). The existence of \( FrGBT D(3, g^u) \)s and \( FrGBT D(4, 4^u) \)s have been solved.

**Theorem 1** [1] Let \( g, u \) be positive integers with \( g \equiv 0 \pmod{3} \) and \( u \geq 5 \). Then there exists an \( FrGBT D(3, g^u) \) with at most 5 possible exceptions of \((g, u) \in \{(6, 15), (9, 18), (9, 28), (9, 34), (30, 15)\} \).

**Theorem 2** [2] Let \( Q_0 = \{19, 21 - 28, 30, 32 - 35, 38 - 40, 45, 47\} \), then for any integer \( u \geq 6 \) and \( u \not\in Q_0 \), there exists an \( FrGBT D(4, 4^u) \).

In this paper, we investigate the existence of \( FrGBT D(4, 4^u) \) for \( g \equiv 0 \pmod{4} \) and \( u \geq 6 \). The sufficiency of the conditions is proved with some possible exceptions.