Factors and Connected Induced Subgraphs

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Abstract. Let $G$ be a connected graph without loops and without multiple edges, and let $p$ be an integer such that $0 < p < |V(G)|$. Let $f$ be an integer-valued function on $V(G)$ such that $2 \leq f(x) \leq \deg_G(x)$ for all $x \in V(G)$. We show that if every connected induced subgraph of order $p$ of $G$ has an $f$-factor, then $G$ has an $f$-factor, unless $\sum_{x \in V(G)} f(x)$ is odd.

1. Introduction

In this paper, we consider only finite undirected graphs. Let $G$ be a graph. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of $G$, respectively. The order of $G$ is denoted by $|G|$. For disjoint subsets $A$ and $B$ of $V(G)$, we let $E(A,B)$ denote the set of edges joining $A$ and $B$, and let $e(A,B)$ denote the cardinality of $E(A,B)$. A vertex $x$ is often identified with $\{x\}$; for example, when $x \notin B$, we write $E(x,B)$ for $E(\{x\},B)$. Also a subgraph $H$ of $G$ is often identified with $V(H)$; in particular, we write $G - H$ for $G - V(H)$. For $x \in V(G)$, we denote by $\deg_G(x)$ and by $N_G(x)$ the degree of $x$ in $G$ and the set of the vertices adjacent to $x$ in $G$; thus if $G$ has no loops and no without multiple edges, $\deg_G(x) = |N_G(x)|$.

For a connected graph $G$, a vertex $x$ is a cutvertex of $G$ if $G - x$ is disconnected. We call $G$ separable if $G$ has a cutvertex, and nonseparable if it has no cutvertex. When $G$ is separable, a maximal nonseparable subgraph of $G$ is called a block of $G$, and a block of $G$ which contains exactly one cutvertex of $G$ is called an endblock of $G$.

Let $g$ and $f$ be integer-valued functions defined on $V(G)$ such that $g(x) \leq f(x)$ for all $x \in V(G)$. A spanning subgraph $F$ of $G$ such that $g(x) \leq \deg_F(x) \leq f(x)$ for all $x \in V(G)$ is called a $(g,f)$-factor of $G$. If $g(x) = f(x)$ for all $x \in V(G)$, a $(g,f)$-factor is called an $f$-factor. Let $k \geq 1$ be an integer. If $f(x) = k$ for all $x \in V(G)$, an $f$-factor is called a $k$-factor. Throughout the rest of this paper, when we say that $G$ is a multigraph, we allow $G$ to have loops and multiple edges, and when we say that $G$ is a graph, we assume that $G$ has no loops and no multiple edges.
Egawa et al. [1] proved the following theorems:

**Theorem A.** Let $G$ be a multigraph, and $p$ be an integer such that $0 < p < |G|$. Let $g,f$ be integer-valued functions defined on $V(G)$ such that $0 \leq g(x) \leq f(x) \leq \deg_G(x)$ for all $x \in V(G)$. Suppose that every induced submultigraph of order $p$ of $G$ has a $(g,f)$-factor. Then $G$ has a $(g,f)$-factor unless $g(x) = f(x)$ for all $x \in V(G)$ and $\sum_{x \in V(G)} f(x)$ is odd.

**Theorem B.** Let $G$ be a connected multigraph with no loops, and let $k$ and $p$ be positive integers such that $0 < p < |G|$ and $k|G|$ is even. Suppose that $G - H$ has a $k$-factor for each connected induced subgraph $H$ of order $p$. Then $G$ has a $k$-factor.

In this paper, we prove the following result related to the above theorems.

**Theorem 1.** Let $G$ be a connected graph, and $p$ be an integer such that $0 < p < |G|$. Let $f$ be an integer-valued function on $V(G)$ such that $2 \leq f(x) \leq \deg_G(x)$ for all $x \in V(G)$. Suppose that every connected induced subgraph $H$ of order $p$ of $G$ has an $f$-factor. Then $G$ has an $f$-factor unless $\sum_{x \in V(G)} f(x)$ is odd.

We prove several preliminary results in Section 2. In Section 3, we prove Theorem 1 and discuss the necessity of the assumption that $f(x) \geq 2$ for all $x \in V(G)$.

2. Graphs with no $f$-Factor

The following criterion for the existence of an $f$-factor is essential for our proof:

**Theorem C (Tutte [2]).** Let $G$ be a graph, and let $f$ be an integer-valued function on $V(G)$ such that $0 \leq f(x) \leq \deg_G(x)$ for all $x \in V(G)$. Then $G$ has an $f$-factor if and only if

$$\delta_G(S,T) := \sum_{x \in S} f(x) + \sum_{y \in T} (\deg_{G-S}(y) - f(y)) - h_G(S,T) \geq 0$$

for all disjoint subsets $S$ and $T$ of $V(G)$, where $h_G(S,T)$ denotes the number of components $C$ of $G - S - T$ such that $e(T, V(C)) + \sum_{z \in V(C)} f(z) \equiv 1 \pmod{2}$.

We also make use of the following lemma:

**Lemma 1 (Tutte [2]).** Under the notation of Theorem C,

$$\delta_G(S,T) \equiv \sum_{x \in V(G)} f(x) \pmod{2}$$

for all disjoint subsets $S$ and $T$ of $V(G)$.

Throughout the rest of this section, we let $G$ be a connected graph, and let $f$ be an integer-valued function on $V(G)$ such that $2 \leq f(x) \leq \deg_G(x)$ for all $x \in V(G)$, $\sum_{x \in V(G)} f(x)$ is even, and $G$ has no $f$-factor. By Theorem C, there exist disjoint subsets $S$ and $T$ of $V(G)$ such that $\delta_G(S,T) < 0$. Then by Lemma D,