Large Monotone Paths in Graphs with Bounded Degree

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Abstract. We prove that for every $\epsilon > 0$ and positive integer $r$, there exists $\Delta_0 = \Delta_0(\epsilon)$ such that if $\Delta > \Delta_0$ and $n > n(\Delta, \epsilon, r)$ then there exists a packing of $K_n$ with $\lfloor (n-1)/\Delta \rfloor$ graphs, each having maximum degree at most $\Delta$ and girth at least $r$, where at most $\epsilon n^2$ edges are unpacked. This result is used to prove the following: Let $f$ be an assignment of real numbers to the edges of a graph $G$. Let $z(G, f)$ denote the maximum length of a monotone simple path of $G$ with respect to $f$. Let $z(G)$ be the minimum of $z(G, f)$, ranging over all possible assignments. Now let $\alpha_\Delta$ be the maximum of $z(G)$ ranging over all graphs with maximum degree at most $\Delta$. We prove that $\Delta + 1 \geq \alpha_\Delta \geq (1 - o(1))$. This extends some results of Graham and Kleitman [6] and of Calderbank et al. [4] who considered $z(K_n)$.

1. Introduction

All graphs considered here are finite, undirected and have no loops or multiple edges. For the standard terminology used the reader is referred to [3]. An edge-ordered graph is an ordered pair $(G, f)$, where $G = G(V, E)$ is a graph and $f$ is an assignment of real weights to the edges. A monotone path of length $k$ in $(G, f)$ is a simple path with $k$ edges, and with nondecreasing edge weights. Given a graph $G$ denote by $z(G)$ the minimum over all edge orderings of the maximum length of a monotone path (note that we can assume $f$ is bijective and that the weights are the integers $1, \ldots, |E|$). Denote by $z'(G)$ the minimum over all edge orderings of the maximum length of a monotone trail (in a trail vertices may appear more than once; a simple cycle is also considered a trail in our definition). Clearly, $z(G) \leq z'(G)$.

The problem of estimating $z(K_n)$ was raised first by Chvátal and Komlós [5]. Graham and Kleitman [6] proved that:

$$\frac{1}{2} \left( \sqrt{4n-3} - 1 \right) < z(K_n) < \frac{3}{4} n.$$

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The upper bound was improved by Calderbank et al. [4], showing,

\[ z(K_n) \leq \left( \frac{1}{2} + o(1) \right) n. \]

They also conjectured that this is the right order of magnitude of \( z(K_n) \). However, no improvement upon the Graham-Kleitman lower bound is known.

There are very few results regarding \( z(G) \) for general graphs \( G \). Bialostocki and Roditty [2] have characterized all the graphs \( G \) with \( z(G) \leq 2 \). In fact, they showed that if \( z(G) \geq 3 \) then either \( G \) is an odd cycle of length at least 5, or \( G \) contains as a subgraph one of six fixed graphs. Roditty et al. gave upper and lower bounds for \( z(G) \) and \( z'(G) \) for graphs \( G \) belonging to several well-known graph families.

In this paper we consider graphs with maximum degree \( \Delta \) and ask how large can \( z(G) \) be. It is very easy to show that \( z(G) \leq \Delta + 1 \) (cf. Lemma 3.1). We may therefore define \( z_\Delta \) as the maximum value of \( z(G) \) taken over all graphs with maximum degree \( \Delta \). Trivially, \( z_1 = 1 \) and \( z_2 = 3 \) (every odd cycle with 5 or more vertices shows this). Despite the \( \Delta + 1 \) upper bound, it is not easy to construct a matching lower bound. In this paper we prove that \( z_\Delta \geq \Delta(1 - o(1)) \). To summarize, we have:

**Theorem 1.1.**

\[ \Delta + 1 \geq z_\Delta \geq \Delta(1 - o(1)). \]

The proof of Theorem 1.1 is based, together with some additional ideas, on a general result concerning packings of complete graphs. Recall that a packing of \( K_n \) is a collection of graphs, sharing the same vertex set of \( K_n \), and which are edge-disjoint. A decomposition of \( K_n \) is a packing that uses every edge of \( K_n \). Recall that the girth of a graph \( G \), denoted \( g(G) \) is the length of the smallest cycle in \( G \). Given a parameter \( \Delta \), and assuming \( \Delta \) divides \( n - 1 \), it is well-known for which values of \( \Delta \) it is possible to decompose \( K_n \) into \( \Delta \)-regular graphs. It merely follows from the classic fact that if \( n \) is odd then \( K_n \) can be decomposed into Hamiltonian cycles, and if \( n \) is even \( K_n \) has chromatic index \( n - 1 \) (see, e.g., [3]). However, the graphs in such a decomposition may (and sometimes will) have small cycles. If we insist that the graphs in the decomposition have girth at least \( r \), we must have \( n \) depend upon \( \Delta \). This, however, is not sufficient. There are examples of pairs \( \Delta \) and \( r \), and arbitrary large \( n \), where \( \Delta \) divides \( n - 1 \), but it is impossible to decompose \( K_n \) into \( \Delta \)-regular graphs with girth at least \( r \). Thus, the best we could hope for is to pack \( K_n \) with \( \lfloor (n - 1)/\Delta \rfloor \) graphs \( H_1, \ldots, H_t \), where \( \Delta(H_i) \leq \Delta \) and \( g(H_i) \geq r \) for \( i = 1, \ldots, t \). Furthermore, at most \( en^2 \) edges of \( K_n \) are unpacked.

**Theorem 1.2.** Let \( \epsilon > 0 \) and let \( r \geq 3 \) be a positive integer. There exists \( \Delta_0 = \Delta_0(\epsilon) \) such that for every \( \Delta > \Delta_0 \) and for every \( n > N(\Delta, \epsilon, r) \), the complete graph \( K_n \) can be packed with \( t = \lfloor (n - 1)/\Delta \rfloor \) graphs \( H_1, \ldots, H_t \), where \( \Delta(H_i) \leq \Delta \) and \( g(H_i) \geq r \) for \( i = 1, \ldots, t \). Furthermore, at most \( \epsilon n^2 \) edges of \( K_n \) are unpacked.