On the Oberwolfach Problem with Two Similar Length Cycles

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Abstract. For all $m \geq 3$, the Oberwolfach problem is solved for the case where the 2-factors consist of two cycles of lengths $m$ and $m + 1$, and for the case where the 2-factors consist of two cycles of lengths $m$ and $m + 2$.

An $(m_1, m_2, \ldots, m_t)$-2-factor of a graph is a 2-factor consisting of cycles of lengths $m_1, m_2, \ldots, m_t$ and an $(m_1, m_2, \ldots, m_t)$-2-factorization of a graph $G$ is a partition of the edge set of $G$ into $(m_1, m_2, \ldots, m_t)$-2-factors. Suppose $n = m_1 + m_2 + \cdots m_t$.

The problem of determining whether there exists an $(m_1, m_2, \ldots, m_t)$-2-factorization of $K_n$ (the complete graph on $n$ vertices) when $n$ is odd, or $K_n - F$ (the complete graph on $n$ vertices with a 1-factor $F$ removed) when $n$ is even, is the Oberwolfach problem, denoted OP$(m_1, m_2, \ldots, m_t)$.

The Oberwolfach problem was formulated by Ringel in 1967 and was first mentioned in [4]. Since then a number of results on the problem and variations of it have appeared in the literature. It is known [2, 3, 5] that OP$(m, m, \ldots, m)$ (arbitrarily many $m$’s) has a solution for all $m \geq 3$ except that there is no solution for OP$(3,3)$ and OP$(3,3,3)$. Many results on the cases where not all the cycles have the same length also exist, see [6–8]. It is known that OP$(4,5)$ and OP$(3,3,5)$ have no solution [8]. A survey of these results can be found in [1].

In this paper, the Oberwolfach problems OP$(m, m + 1)$ and OP$(m, m + 2)$ are solved for all integers $m \geq 3$ (see Theorems 1, 2, 3 and 4).

We will use the following notation. An $m$-cycle, denoted $C_m$, is a graph $(v_1, v_2, \ldots, v_m)$ with vertex set $\{v_1, v_2, \ldots, v_m\}$ and edge set $\{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_m, v_1\}\}$. Let $G^c$ denote the complement of the graph $G$. If $G$ and $H$ are two graphs, then let $G \cup H$ be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$, and if $V(G) \cap V(H) = \emptyset$ then let $G \vee H$ be the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H) \cup \{\{g, h\}|g \in V(G), h \in V(H)\}$.

Lemma 1. Let $m \geq 4$ be even. Then there is an $(m, m + 2)$-2-factorization of $K_m^c \cup C_{m+2}$.

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Proof. Let $n \in \{m, m+2\}$ with $n \equiv 2 \pmod{4}$, let $C_{m+2} = (b_1, b_2, \ldots, b_{m+2})$ and let $V(K_m) = \{a_1, a_2, \ldots, a_m\}$. Then the orbit of the 2-factor shown in Figure 1 under the permutation $(b_1, b_3, \ldots, b_{m+1}, b_2, b_4, \ldots, b_{m+2})$ gives the required 2-factorization.

\[ \Box \]

**Theorem 1.** Let $m \geq 4$ be even. Then there is an $(m, m+2)$-2-factorization of $K_{2m+2} - F$.

**Proof.** Let $A$ be the complete graph of order $m$ with $V(A) = \{a_1, a_2, \ldots, a_m\}$, let $B$ be the complete graph of order $m+2$ with $V(B) = \{b_1, b_2, \ldots, b_{m+2}\}$ with $V(A) \cap V(B) = \emptyset$ and let $V(K_{2m+2} - F) = V(A) \cup V(B)$. Form $(m-2)/2$ $(m, m+2)$-2-factors in $K_{2m+2} - F$ by pairing off the $m$-cycles in an $(m)$-2-factorization of $A - F'$ (where $F'$ is a 1-factor in $A$) with all but one of the $(m+2)$-cycles in an $(m+2)$-2-factorization of $B - F''$ (where $F''$ is a 1-factor in $B$). Letting $F = F' \cup F''$, this leaves $K_m \setminus C_{m+2}$ which has an $(m, m+2)$-2-factorization by Lemma 1. □

**Theorem 2.** Let $m \geq 3$ be odd. Then there is an $(m, m+2)$-2-factorization of $K_{2m+2} - F$.

**Proof.** Let $V(K_{m+2}) = \{\infty_1, \infty_2, a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m\}$ and let $F = \{\{\infty_1, \infty_2\}, \{a_1, b_{(m+3)/2}\}, \{a_2, b_{(m+5)/2}\}, \ldots, \{a_{(m-1)/2}, b_m\}, \{a_{(m+1)/2}, b_1\}, \ldots, \{a_m, b_{(m+1)/2}\}\}$. Then the orbit of the 2-factor shown in Figure 2 under the permutation $(a_1, a_2, \ldots, a_m)$ $(b_1, b_2, \ldots, b_m)$ gives the required 2-factorization. □

**Lemma 2.** There is a $(3, 4)$-2-factorization of $K_7$ and a $(5, 6)$-2-factorization of $K_{11}$.

**Proof.** Let $V(K_7) = \{\infty, a_1, a_2, a_3, b_1, b_2, b_3\}$. Then the orbit of the 2-factor

$$(a_1, a_2, b_1, b_2) \cup (\infty, a_3, b_3)$$

under the permutation $(a_1, a_2, a_3)(b_1, b_2, b_3)$ gives the required 2-factorization. Let $V(K_{11}) = \{\infty, a_1, a_2, a_3, a_4, a_5, b_1, b_2, b_3, b_4, b_5\}$. Then the orbit of the 2-factor

$$(a_1, a_3, a_4, b_2, a_2, b_3) \cup (\infty, a_5, b_4, b_1, b_5)$$

![Fig. 1](image-url)