On Independent Cycles in a Bipartite Graph

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Abstract. Let $G = (V_1, V_2; E)$ be a bipartite graph with $2k \leq m = |V_1| \leq |V_2| = n$, where $k$ is a positive integer. We show that if the number of edges of $G$ is at least $(2k - 1)(n - 1) + m$, then $G$ contains $k$ vertex-disjoint cycles, unless $e(G) = (2k - 1)(n - 1) + m$ and $G$ belongs to a known class of graphs.

1. Introduction

We discuss only finite simple graphs and use standard terminology and notation from [4] except as indicated. A set of graphs is said to be independent if no two of them have any vertex in common. Corrádi and Hajnal [5] investigated the maximum number of independent cycles in a graph. They proved that a graph $G$ of order $n \geq 3k$ contains $k$ independent cycles provided $\delta(G) \geq 2k$ holds. In particular, when the order of $G$ is exactly $3k$, then $G$ contains $k$ independent triangles. In [11], we considered a similar problem in bipartite graphs. We use $(X, Y; E)$ to denote a bipartite graph with $(X, Y)$ as its bipartition and $E$ as its edge set. A quadrilateral is a cycle of length 4. We proved the following two theorems.

Theorem 1. [11] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = n > 2k$, where $k$ is a positive integer. Suppose that the minimum degree of $G$ is at least $k + 1$. Then $G$ contains $k$ independent cycles.

Theorem 2. [11] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$, where $k$ is a positive integer. Suppose that the minimum degree of $G$ is at least $k + 1$. Then $G$ contains $k - 1$ independent quadrilaterals and a path of order 4 such that the path is independent of all the $k - 1$ quadrilaterals.

Moreover, it is also shown in [11] that the conditions on degrees of $G$ in the above theorems are sharp. We conjectured the following.

Conjecture 3. [11] Let $G = (V_1, V_2; E)$ be a bipartite graph with $|V_1| = |V_2| = 2k$, where $k$ is a positive integer. If the minimum degree of $G$ is at least $k + 1$, then $G$ contains $k$ independent quadrilaterals.
Following the above work, we now consider the conditions on the size of a bipartite graph containing \( k \) independent cycles. We say that a bipartite graph has an \((a, b)\)-bipartition if it has a bipartition \((X, Y)\) such that \(|X| = a\) and \(|Y| = b\). To state our result, we define the following classes \( \Sigma_{k,m,n} \) of bipartite graphs for all positive integers \( k, m \) and \( n \) with \( 2k \leq m \leq n \).

If \( k = 1 \), then \( \Sigma_{1,m,n} \) contains all trees with an \((m, n)\)-bipartition. If \( k \geq 2 \), then a bipartite graph \( G \) belongs to \( \Sigma_{k,m,n} \) if and only if there exist an \((m, n)\)-bipartition \((X, Y)\) of \( G \) and a subset \( Z \) of \( X \) with \(|X| = m\), \(|Y| = n\) and \(|Z| = 2k - 1\) such that the subgraph of \( G \) induced by \( Y \cup Z \) is a complete bipartite graph (i.e., isomorphic to \( K_{2k-1,m} \)) and each vertex in \( X - Z \) has degree one in \( G \). It is easy to see that each graph in \( \Sigma_{k,m,n} \) has \((2k - 1)(n - 1) + m\) edges but does not have \( k \) independent cycles. In this paper, we prove the following theorem.

**Theorem 4.** Let \( G = (V_1, V_2; E) \) be a bipartite graph with \( 2k \leq m = |V_1| \leq |V_2| = n \), where \( k \) is a positive integer. Suppose that the number of edges of \( G \) is at least \((2k - 1)(n - 1) + m\) and \( G \) does not belong to \( \Sigma_{k,m,n} \). Then \( G \) contains \( k \) independent cycles.

As for general graphs, Andreade and Justesen [3, 8] found the conditions on the size of a graph \( G \) of order \( n \) that ensure the existence of \( k \) independent cycles in \( G \). As pointed out in [3], the arguments in [3, 8] are based on the result of Corrádi and Hajnal [5]. Unlike this, our proof of Theorem 4 is independent of Theorem 1 and Theorem 2. For related results, see [1, 2].

We need the following notation and terminology. Let \( G = (V, E) \) be a graph. The number of edges of \( G \) is denoted by \( e(G) \). For any \( u \in V \), if \( G' \) is a subgraph of \( G \) or a subset of \( V \), we define \( N(u, G') \) to be the set of neighbours of \( u \) which are contained in \( G' \) and let \( d(u, G') = |N(u, G')| \). Thus \( d(u, G) = d(u, V) = |N(u)| \) is the degree \( d(u) \) of \( u \) in \( G \). If \( d(u, G) = 1 \) we say that \( u \) is an endvertex of \( G \). For a subset \( U \) of \( V \), \( G[U] \) is the subgraph of \( G \) induced by \( U \). For two independent subgraphs \( G_1 \) and \( G_2 \) of \( G \), \( e(G_1, G_2) \) is the number of edges of \( G \) between \( G_1 \) and \( G_2 \), i.e., \( e(G_1, G_2) = \sum_{x \in F(G_1)} d(x, G_2) \). For a positive integer \( t \), if we write \( G \supseteq \mathcal{O} \), it means that \( G \) contains a set of \( t \) independent cycles, and by \( G \supseteq k\mathcal{K}_{2,2} \), it means that \( G \) contains a set of \( t \) independent quadrilaterals. If \( G \) is a cycle, its length will be denoted by \( l(G) \).

2. Lemmas

In what follows, \( G = (V_1, V_2; E) \) is a given bipartite graph. The following lemmas except for Lemma 2.4 are adopted from [11].

**Lemma 2.1.** [11] Let \( C \) be a cycle of \( G \) and \( x \) a vertex of \( G \) not on \( C \). Suppose \( d(x, C) \geq 2 \). Then either \( C \) is a quadrilateral or \( C + x \) contains a cycle \( C' \) such that \( l(C') < l(C) \).