The Diameters of Some Transition Graphs Constructed from Hamilton Cycles

Mariko Hagita\textsuperscript{1*}, Yoshiaki Oda\textsuperscript{2†}, and Katsuhiro Ota\textsuperscript{3}

Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kohoku-ku, Yokohama 223-8522, Japan
e-mails: \textsuperscript{1}hagita, \textsuperscript{2}oda, \textsuperscript{3}ohta@comb.math.keio.ac.jp

Abstract. A complete undirected graph of order \(n\) has \(\frac{(n-1)!}{2}\) Hamilton cycles. We consider the diameter of a transition graph whose vertices correspond to those Hamilton cycles and any of two vertices are adjacent if and only if the corresponding Hamilton cycles can be transformed each other by exchanging two edges. Moreover, we consider several transition graphs related to it.

1. Motivations

The Traveling Salesman Problem (TSP) is the problem of finding a shortest tour in a complete weighted graph of order \(n\). The TSP is one of the most famous NP-hard problems. So, many works have been done to find algorithms to get a nearly optimal tour. The 2-OPT is one of the most famous local search methods for the TSP.

Definition 1.1. For given \(n\) vertices and a tour \(T\), if the tour \(T'\) is constructed from \(T\) by deleting two edges and adding other two edges, the transformation from \(T\) to \(T'\) is called a 2-change. The 2-change from \(T\) to \(T'\) such that the total length of \(T'\) is less than the one of \(T\) is called a feasible 2-change. The procedure repeating feasible 2-change operations until there is no feasible 2-change is called 2-OPT.

The 2-OPT is one of good local search methods, but the following negative fact is known.

Theorem 1.2 (Lueker (see [2])). There exist instances which must need exponential times of feasible 2-change operations to reach a local optimal solution.

So, we consider the following problem.

\* Current address: Faculty of Environmental Information, Keio University, 5322 Endo, Fujisawa 252-8520, Japan. e-mail: hagita@sfc.keio.ac.jp
\† Current address: Department of Mathematics and Computer Science, Shimane University, 1060 Nishikawatsu-cho, Matsue 690-8504, Japan. e-mail: oda@cis.shimane-u.ac.jp
Problem 1.3. We discard the condition on the length of tours. How many 2-change operations do we need to transform one Hamilton cycle to another?

We define the transition graph $C^*_n$, as mentioned in the next section. The diameter of $C^*_n$ corresponds to the maximum number of times of 2-change operations for this problem.

2. Definitions and Backgrounds

Definition 2.1. The transition graph $C^*_n$ is the graph whose vertices correspond to all Hamilton cycles of a complete graph of order $n$, and two vertices are adjacent if and only if the corresponding two Hamilton cycles can be transformed each other by exchanging two edges, that is, removing two edges and adding other two edges.

Suppose that the vertices of a complete graph of order $n$ are labelled from 1 to $n$. In this paper, a Hamilton cycle is denoted by $(i_1, \ldots, i_n)$ and a permutation is denoted by $(i_1, \ldots, i_n)$. For example, if the Hamilton cycle $\sigma = (c_1, \ldots, c_n)$ changes into $\sigma' = (c_1, \ldots, c_{i-1}, c_j, c_{j-1}, \ldots, c_i, \ldots, c_n)$ by deleting two edges $(c_{i-1}, c_i)$ and $(c_j, c_{j+1})$ and adding $(c_{i-1}, c_j)$ and $(c_i, c_{j+1})$, then the corresponding vertices in $C^*_n$ are adjacent. This exchange operation can be considered to reverse a subpath (subsequence) from $c_i$ to $c_j$ with length at least two of a Hamilton cycle. We call such an operation an inversion for $\sigma$ and denote it by $\rho = [i, j]$. We also denote the resulting Hamilton cycle $\sigma'$ by $\sigma \rho$. We note that if $\rho = [i, j]$ with $i > j$ then $\sigma \rho$ can be considered as $\sigma[j+1, i-1]$. A neighbouring inversion is the inversion which reverses a subpath with length two. Not to mention that a Hamilton cycle $(c_1, c_2, \ldots, c_n)$ is identical with $(c_2, \ldots, c_n, c_1)$ and $(c_n, c_{n-1}, \ldots, c_1)$ etc., but we note that an inversion is defined for the cycle with a fixed orientation and a fixed starting vertex, that is, in general, the inversion $[i, j]$ of $(c_1, \ldots, c_n)$ is different from the inversion $[i, j]$ of $(c_2, \ldots, c_n, c_1)$. We also define the following:

Definition 2.2. The transition graph $C^k_n$ is the graph whose vertices correspond to the Hamilton cycles of a complete graph of order $n$, and two vertices are adjacent if and only if the corresponding two Hamilton cycles can be transformed each other by an inversion of a subpath with length $k$.

![Fig. 1. The vertex corresponding to the Hamilton cycle (1,2,3,4,5,6) and its neighbourhood on the transition graph $C^2_n$.](image-url)