A Class of 2-Colorable Orthogonal Double Covers of Complete Graphs by Hamiltonian Paths

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Abstract. An orthogonal double cover (ODC) of the complete graph $K_n$ by a graph $G$ is a collection $\mathcal{G}$ of $n$ spanning subgraphs of $K_n$, all isomorphic to $G$, such that any two members of $\mathcal{G}$ share exactly one edge and every edge of $K_n$ is contained in exactly two members of $\mathcal{G}$. In the 1980’s Hering posed the problem to decide the existence of an ODC for the case that $G$ is an almost-hamiltonian cycle, i.e. a cycle of length $n - 1$. It is known that the existence of an ODC of $K_n$ by a hamiltonian path implies the existence of ODCs of $K_{2n}$ and $K_{2n+1}$, respectively, by almost-hamiltonian cycles. Horton and Nonay introduced 2-colorable ODCs and showed: If for $n \geq 3$ and a prime power $q \geq 5$ there are an ODC of $K_n$ by a hamiltonian path and a 2-colorable ODC of $K_q$ by a hamiltonian path, then there is an ODC of $K_{qn}$ by a hamiltonian path. We construct 2-colorable ODCs of $K_n$ and $K_{2n}$, respectively, by hamiltonian paths for all odd square numbers $n \geq 9$.

Key words. Orthogonal double cover, Graph decompositions, Self-orthogonal decompositions, Self-orthogonal 2-sequencings, Hering’s problem

1. Introduction

Motivated by questions from design theory [26] and from the theory of data bases [10], the intensive study of orthogonal double covers of complete graphs started in the 1980’s. Since then there has been a steadily growing interest in the subject.

An orthogonal double cover (ODC) of the complete graph $K_n$ is a collection $\mathcal{G}$ of $n$ spanning subgraphs of $K_n$ such that the following two conditions are satisfied:

(i) Every two distinct members of $\mathcal{G}$ share exactly one edge.
(ii) Every edge of $K_n$ is contained in exactly two members of $\mathcal{G}$.

Every $G \in \mathcal{G}$ is called a page of $\mathcal{G}$. The above definition immediately implies $|E(G)| = n - 1$ for all $G \in \mathcal{G}$. If all members of $\mathcal{G}$ are isomorphic to some graph $G$, then $\mathcal{G}$ is called an ODC of $K_n$ by $G$.

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The principle question concerning ODCs is: Given some graph $G$ with $|E(G)| = n - 1$, is there an ODC of $K_n$ by $G$? This question has been investigated for several classes of graphs such as graphs consisting of short cycles [10, 13, 14], almost-hamiltonian cycles [27, 26, 2, 25, 24], graphs with maximum degree 2 [9, 18, 7], clique graphs [6,12], and trees [16, 29, 30], in particular hamiltonian paths [24, 28, 3, 4, 31, 21]. Most of the constructions use group-actions or $PBD$-closure (see [5]). The notion of an ODC can be generalized in various directions, for instance the directed analogue of the problem [27, 2, 24, 28, 15, 20], suborthogonal double covers [19, 33, 22], and symmetric graph designs [17, 8, 11] have been considered.

Here we focus on ODCs by hamiltonian paths and almost-hamiltonian cycles.

2. Group-Generated ODCs

In constructing ODCs it turned out to be fruitful to generate ODCs using the action of some group. Many of the known classes of ODCs are group-generated.

Let $\mathcal{A}$ be an additive abelian group of finite order $n$, and let $G$ be a graph with $V(G) \subseteq \mathcal{A}$. Furthermore, for $x \in \mathcal{A}$ let $G + x$ denote the graph defined by

$$V(G + x) = \{y + x : y \in V(G)\}, \quad E(G + x) = \{\{y + x, z + x\} : \{y, z\} \in E(G)\}.$$ 

For every $x \in \mathcal{A}$ the graph $G + x$ is called a translate of $G$. If the set

$$G + \mathcal{A} := \{G + x : x \in \mathcal{A}\}$$

is an ODC of $K_n$, then we say that this ODC is generated by $\mathcal{A}$.

To formulate necessary and sufficient conditions for $G + \mathcal{A}$ to be an ODC, let us introduce two more notions. The length of an edge $e = \{x, y\} \in E(G)$ is defined by $\ell(e) := \{x - y, y - x\}$, and the distance of two edges $e = \{x, y\} \in E(G)$, $e' = \{x + z, y + z\} \in E(G)$ of the same length is the set $d(e, e') := \{z, -z\}$.

**Lemma 1 (cf. [28]).** Let $\mathcal{A}$ be a finite abelian group, and let $G$ be a graph on the vertex set $\mathcal{A}$. The set $G + \mathcal{A}$ is an ODC if and only if the following three conditions are satisfied:

1. For every $x \in \mathcal{A}$ of order 2 there is exactly one edge $e \in E(G)$ such that $\ell(e) = \{x\}$.
2. For every $x \in \mathcal{A}$ of order $> 2$ there are exactly two distinct edges $e, e' \in E(G)$ such that $\ell(e) = \ell(e') = \{x, -x\}$.
3. $$\bigcup_{e, e' \in E(G), e \neq e', \ell(e) = \ell(e')} d(e, e') = \{x \in \mathcal{A} : \text{ord}(x) > 2\}$$