The Concept of Primitivity in Group Theory and the Second Memoir of Galois

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1. Introduction

The so-called Second Mémoire of Évariste Galois, written in 1830 and first published in 1846, is notoriously difficult to understand. Already the title ‘Des équations primitives qui sont solubles par radicaux’ requires considerable thought. For, the word ‘primitive’, which has a standard meaning in the context of group theory now, had no such context or meaning when Galois used it.

I have two goals in the present paper: first, to give an account of the development of the concept of primitivity in finite group theory in the 19th century; secondly, to provide commentary on the first part of the Second Mémoire. These may appear to be two separate projects for which two separate papers would be appropriate. I combine them into one because, as the reader will see, they are in fact so closely related as to be almost inseparable. Hence my choice of an ambiguous title.

I also have two categories of reader in mind: historians and mathematicians. For the former, but also because one cannot discuss the history of the mathematics without having the details in front of one, I give expositions of some basic theory of equations and groups. For the latter I rehearse some of the known facts about Galois’ writings.

Almost certainly the Second Mémoire was written in 1830 (see [18, p. 494]). It was first published in 1846 by Liouville [15] and there have been several re-publications since, culminating in the critical edition of 1962 by Robert Bourgne and J.-P. Azra [18]. It is an unfinished first draft, nowhere near as much revised or as important as the Premier Mémoire. Nevertheless, where the Premier Mémoire describes what we now think of as Galois Theory, the Second Mémoire focusses heavily on groups and, as we shall see, has had, through the work of Camille Jordan and others, an important influence on group theory. It provides moreover a significant piece of evidence about the workings of the mind of an extraordinarily creative and intuitive young genius. Either one of these facts would be reason to make it worthy of intensive study; together they are compelling.

The Second Mémoire is in two parts. The first is about finite soluble equations and groups, the second about the groups we now call AGL(2, p) and PSL(2, p). The present paper is concerned only with the first part—I propose to write about the second part on some other occasion.
That first part contains Galois’ wonderful insight (shared, as we shall see, by N. H. Abel) that a primitive soluble equation has prime-power degree. Equivalently:

*a finite primitive soluble permutation group has prime-power degree.*

His insight goes deeper than this—in his famous testamentary letter to Auguste Chevalier written on 29 May 1832, in the night before his fatal duel, he explains correctly that (in modern language) if the primitive soluble group has degree $p^n$ then it may be thought of as consisting of affine transformations of the form $v \mapsto vA + b$ of a $v$-dimensional vector space $V$ over the field $\mathbb{F}_p$ of integers modulo $p$. (Here $A$ is an invertible $v \times v$ matrix over $\mathbb{F}_p$ and $b \in V$.) Although his insight is correct, and the theorems are quite as important as he believed them to be, the arguments he gives in the *Second Mémoire* are not easy to understand. Indeed, very probably they are wrong. To be certain that they are wrong, and cannot be corrected or completed, one would need to be sure what Galois meant when he described an equation as being primitive. And that, as has already been said, is the subject of this essay.

The idea of primitivity is fundamental in permutation group theory. My purpose is not only to seek to elucidate what Galois could have had in mind, but also to trace its development to its modern form, which had been reached by 1870, the year when Camille Jordan’s great *Traité des Substitutions et des Équations algébriques* [26] appeared. As we shall see, there are two concepts which might naturally be taken to be what Galois intended. These merge in an interesting way in Jordan’s doctoral thesis of 1860/61. A few years later Jordan had established not only the modern notion but also the main reasons for its importance.

One of our difficulties *amice lector* will be to distinguish Galois’ use of the word *primitif* (usually occurring in the form *primitive* as an adjective qualifying the feminine noun *équation*) from my use of the modern technical term ‘primitive’ as it is used in the theory of permutation groups. Throughout the text of this paper, but excluding the quotations, italics will be used, as above, for Galois’ French word and roman type for the modern one. Another minor difficulty in quotations will be with such usages as ‘Oeuvres’, ‘Œuvres’, or ‘de Evariste’, or ‘d’Evariste’, or ‘que une’ and ‘qu’une’. In all such cases I have chosen the line of least resistance and sacrificed consistency for faithfulness to the original.

This paper is structured as follows. Section 2 contains an analysis of what Galois meant by his use of the word *primitif* in the context of polynomials and equations, and §3 correlates this with the modern notions of primitivity and quasiprimitivity in group theory. In §4 I discuss the problems associated with translating and interpreting the *Second Mémoire* (and other works of Galois). Section 5 is devoted to the main source material. It contains, paragraph by paragraph, a transcription, a translation into English, and a critique of the *Second Mémoire*. In §§6, 7 we examine relevant passages from the works of N. H. Abel (1826/1881) and A.-L. Cauchy (1845). From 1860 to 1870 the notion of primitivity was developed by Camille Jordan and his contributions are analysed in §§8, 9. The subject of §10 is the establishment of primitivity as a basic notion of group theory. Section 11 contains brief comment on commentators of the *Second Mémoire* and §12 contains a summary of my conclusions. My analysis of the *Second Mémoire*