Boundary contour analysis for surface stress recovery in 2-D elasticity and Stokes flow

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Summary A variant of the boundary element method, called the boundary contour method, offers a further reduction in dimensionality. Consequently, boundary contour analysis of 2-D problems does not require any numerical integration at all. In a boundary contour analysis, boundary stresses can be accurately computed using the approach proposed in Ref. [1]. However, due to singularity, this approach can be used only to calculate boundary stresses at points that do not lie at an end of a boundary element. Herein, it is shown that a technique based on the displacement/velocity shape functions can overcome this drawback. Further, the approach is much simpler to apply, requires less computational effort, and provides competitive accuracy. Numerical solutions and convergence study for some well-known problems in linear elasticity and Stokes flow are presented to show the effectiveness of the proposed approach.

Keywords Boundary contour method, Boundary element method, Stress analysis, Stress recovery

1 Introduction
The primary advantage of the boundary element method (BEM) over domain methods such as finite difference and finite element methods is that the BEM requires discretizing only the boundary of the region of interest. As a result, the BEM can be particularly effective in solving moving-boundary problems including shape optimization [2], crystal growth [3], or fluid motion [4, 5] as repeated remeshing of a moving boundary is much less expensive than repeated remeshing of its domain. For this class of application, boundary stress is usually the primary parameter of interest, as it could be used to determine the velocity of the moving boundary. Thus, both accurate and fast recovery of boundary stresses are equally important issues. In the context of the BEM, considerable efforts have been devoted to developing effective algorithms for recovering (post-processing evaluating) boundary stresses (e.g., [6–11]). For a comprehensive review on this topic and additional references to the literature, the reader is referred to Ref. [8].

The conventional BEM for linear elasticity requires the numerical evaluation of line integrals for 2-D problems and surface integrals for 3-D ones (e.g., [12, 13]). By observing that the integrand vector of the boundary integral equation (BIE) for the Laplace equation is divergence...
free, Lutz [14] has shown that a further reduction in dimensionality can be achieved. Nagarajan et al. [15, 16] have extended the idea to linear elasticity, and a numerical implementation of the approach is termed BCM. The divergence free property allows, for 3-D problems, the use of Stokes’ theorem to transform surface integrals on the usual boundary elements into line integrals on the bounding contours of these elements (thus, the name boundary contour method). For 2-D problems, a transformation based on this divergence-free property converts line integrals to path-independent integrals, which do not require any numerical integration: integrals are evaluated using potential functions in closed form. The above transformations are quite general and apply to boundary elements of arbitrary shapes. Thus, the BCM requires only numerical evaluation of line integrals for 3-D problems and simply the evaluation of potential functions at points on the boundary of a body for 2-D cases. The BCM also works for other linear problems such as potential theory [17], or Stokes flow [18]. In addition to the aforementioned BCM work, numerous papers have been devoted to the development and application of the method. The primary development work reported in the literature has been for 2-D [1, 15, 19–22] and for 3-D [16, 23, 24] linear elasticity problems. The method has also been successfully applied to design sensitivity analysis [2, 25–28], shape optimization [2, 29, 30], analysis of thin films and layered coatings [31], and fracture mechanics [32, 33].

As with the BEM, boundary stress recovery in the context of the BCM can be done by attacking the hypersingular boundary integral equation (HBIE) [19] or by differentiating displacement/velocity shape functions. In addition, boundary stress can also be accurately post-processed by differentiating the BCM potential functions [1]. The primary advantages of the BCM over the conventional BEM in evaluating boundary stresses are that there are no numerical integrations involved for 2-D problems, and no special treatment of singularity is required. In this paper, we present an approach to calculation of boundary stresses for 2-D linear elasticity and Stokes flow through the use of gradients of the shape functions. Since shape functions employed in the BCM are expressed in terms of Cartesian coordinates, it is more straightforward and requires much less computational effort to obtain the displacement/velocity gradients than for the BEM. Also, due to the absence of numerical integration and the use of Cartesian coordinate system, boundary stresses recovered by this approach appear to be competitively accurate.

Comparative and convergence studies are presented for some illustrative Stokes flow and elastic examples, including the flow between two concentric cylinders, Lamé’s and Kirsch’s problems. In all cases, the numerical results are very accurate and converged with respect to the mesh density.

2
2-D BCM formulation
The dimensional reduction starts from the regularized BIE without body forces for elasticity and Stokes flow (e.g., [34] and [35])

\[ 0 = \int_{\partial B} \left\{ U_{ik}(P, Q) t_i(Q) - T_{ik}(P, Q) [u_i(Q) - u_i(P)] \right\} n_j(Q) \, dS, \]

where \( P, Q, u_i, \) and \( t_i \) are source point, field point, displacement/velocity, and traction vectors, respectively, and \( U_{ik} \) and \( T_{ik} \) are the kernel tensors. In 2-D problems, \( \partial B \) is the boundary curve defining the body \( B, \) and \( i, k = 1, 2. \)

By using Cauchy’s (traction-stress) relation

\[ t_i = \sigma_{ij} n_j, \]
\[ T_{ik} = \Sigma_{ijk} n_j, \]

the regularized BIE (1) becomes

\[ 0 = \int_{\partial B} \left\{ U_{ik}(P, Q) \sigma_{ij}(Q) - \Sigma_{ijk}(P, Q) [u_i(Q) - u_i(P)] \right\} n_j(Q) \, dS = \int_{\partial B} G_{ij} n_j(Q) \, dS, \]

where \( n_j \) is the unit outward normal to \( \partial B. \)