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First passage failure of quasi non-integrable generalized Hamiltonian systems

Abstract The first passage failure of quasi non-integrable generalized Hamiltonian systems is studied. First, the generalized Hamiltonian systems are reviewed briefly. Then, the stochastic averaging method for quasi non-integrable generalized Hamiltonian systems is applied to obtain averaged Itô stochastic differential equations, from which the backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time are established. The conditional reliability function and the conditional mean of first passage time are obtained by solving these equations together with suitable initial condition and boundary conditions. Finally, an example of power system under Gaussian white noise excitation is worked out in detail and the analytical results are confirmed by using Monte Carlo simulation of original system.

Keywords Quasi non-integrable generalized Hamiltonian system · Stochastic averaging · First passage failure

1 Introduction

In the past few decades, there has been considerable interest in the study of reliability of stochastic dynamical systems due to its practical significance in determining the performance of a wide range of systems. The first passage is the most important failure model in stochastic dynamics but it is very difficult to obtain the first passage probability. The known exact solution of the first passage problem is limited to the one-dimensional diffusion process. Therefore, several numerical methods such as generalized cell-mapping procedure [1], Monte Carlo simulation [2] and the singular perturbation [3] have been proposed to obtain the statistic of first passage problem for higher-dimensional stochastic systems in the literature. At present, a powerful approximate technique for analyzing the first passage problem of higher-dimensional stochastic systems is the combination approach of the stochastic averaging method and the diffusion process theory of the first passage time, which has been applied by many authors (for example, see [4–9] and the references therein).

In practice, many systems in science and engineering are of odd dimension, which can be modeled as stochastically excited and dissipated generalized Hamiltonian systems. Such systems may be classified into five groups based on the integrability and resonance of the associated generalized Hamiltonian systems. An $n$-dimensional quasi non-integrable generalized Hamiltonian system is a generalized Hamiltonian system
with $M$ Casimir functions $C_1, \ldots, C_M$ and one first integral $H$ subject to lightly linear and (or) nonlinear dampings and weakly random excitations. To the authors’ knowledge, the first passage failure of quasi non-integrable generalized Hamiltonian systems has not been studied so far.

In the present paper, the equations governing quasi non-integrable generalized Hamiltonian system are reduced to a set of averaged Itô stochastic differential equations by using the stochastic averaging method. Then, the backward Kolmogorov equation governing the conditional reliability function and the Pontryagin equation governing the conditional mean of the first passage time are derived from the averaged equations. Finally, a three-dimensional power system subject to Gaussian white noise excitation is taken as an example to illustrate the proposed procedure. The numerical results for the example are verified by using those from Monte Carlo simulation of original system.

2 Generalized Hamiltonian systems

An $n$-dimensional dynamical system governed by

\[ \dot{x}_i = J_{ij}(x) \frac{\partial H'}{\partial x_j}, \quad i, j = 1, \ldots, n \]  

(1)

is called a generalized Hamiltonian system. In Eq. (1), $x = [x_1, \ldots, x_n]^T$ is a state vector; the dot denotes the derivative with respect to time $t$; $H' = H(x)$ is the twice differentiable generalized Hamiltonian; $[J_{ij}(x)]$ is an $n \times n$ anti-symmetric structural matrix, which satisfies the Jacobi identities [10] and, therefore, provides a generalized Poisson bracket

\[ [F, G] = \sum_{i=1}^{n} \sum_{j=1}^{n} \frac{\partial F}{\partial x_i} J_{ij} \frac{\partial G}{\partial x_j} \]  

(2)

for two dynamical quantities $F(x)$ and $G(x)$ in phase space.

A function $F = F(x)$ is called a first integral of system (1) if $[F, H] = 0$. A function $C = C(x)$ is called Casimir function if $[C, G] = 0$, where $G = G(x)$ is any real-valued function. Obviously, a Casimir function is a first integral of the system (usually generalized Hamiltonian systems). A generalized Hamiltonian system having only one first integral except Casimir functions is called the non-integrable generalized Hamiltonian system.

3 Stochastic averaging

Consider a stochastically excited and dissipative generalized Hamiltonian system governed by the following equations:

\[ \dot{X}_i = J_{ij}(X) \frac{\partial H'}{\partial X_j} + \varepsilon d_{ij}(X) \frac{\partial H'}{\partial X_j} + \varepsilon^{1/2} f_{is}(X) W_s(t), \quad i, j = 1, \ldots, n; \quad s = 1, \ldots, l \]  

(3)

where $X = [X_1, \ldots, X_n]^T$; $[J_{ij}(X)]$ is an $n \times n$ anti-symmetric structural matrix; $H'(X)$ is twice differentiable generalized Hamiltonian; $\varepsilon d_{ij}(X)$ and $\varepsilon^{1/2} f_{is}(X)$ are the coefficients of dampings and the amplitudes of stochastic excitations, respectively; $\varepsilon$ is a small positive parameter; $W_s(t)$ are Gaussian white noises with intensities $E[W_s(t)W_s(t+\tau)] = 2D_{sz}\delta(t,\tau), s, z = 1, \ldots, l$.

Equation (3) can be modeled as Stratonovich stochastic differential equations and then converted into Itô stochastic differential equations by adding the Wong-Zakai correction terms $D_{sz} f_{is} f_{iz} / \dot{X}_j$. Splitting the Wong-Zakai correction terms into conservative part and dissipative part, and combining the two parts with $J_{ij}(\dot{X}) \partial H'/\partial X_j$ and $d_{ij}(\dot{X}) \partial H'/\partial X_j$, respectively, Eq. (3) is converted into the following Itô equations:

\[ dX_i = \left[ J_{ij}(X) \frac{\partial H}{\partial X_j} + \varepsilon d_{ij}(X) \frac{\partial H}{\partial X_j} \right] dt + \varepsilon^{1/2} \sigma_{is}(X) dB_s(t), \quad i, j = 1, \ldots, n; \quad s = 1, \ldots, l \]  

(4)

where $[J_{ij}(X)]$ is a modified structural matrix; $H$ is a modified Hamiltonian; $\varepsilon d_{ij}(X)$ is the coefficients of modified dampings; $B_s(t)$ are the standard Wiener processes and $\sigma \sigma^T = 2DF^T$. 

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