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Elasticity solutions for functionally graded annular plates subject to biharmonic loads

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Abstract Based on England’s expansion formula for displacements, the elastic field in a transversely isotropic functionally graded annular plate subjected to biharmonic transverse forces on its top surface is investigated using the complex variables method. The material parameters are assumed to vary along the thickness direction in an arbitrary fashion. The problem is converted to determine the expressions of four analytic functions $\alpha(\zeta)$, $\beta(\zeta)$, $\phi(\zeta)$ and $\psi(\zeta)$ under certain boundary conditions. A series of simple and practical biharmonic loads are presented. The four analytic functions are constructed carefully in a biconnected annular region corresponding to the presented loads, which guarantee the single-valuedness of the mid-plane displacements of the plate. The unknown constants contained in the analytic functions can be determined from the boundary conditions that are similar to those in the plane elasticity as well as those in the classical plate theory. Numerical examples show that the material gradient index and boundary conditions have a significant influence on the elastic field.

Keywords Functionally graded materials · Annular plates · Transversely isotropic · Biharmonic load · Elasticity solutions

1 Introduction

Functionally graded materials (FGMs) are a new type of inhomogeneous materials, which can be used to meet different requirements for material service performance at different locations in structures due to the exhibiting gradient change of macroscopic properties in space. Therefore, a large number of research activities have been directed to the study of elastic responses of FGM plates under various conditions.
There are several methods that have been proposed to analyze the bending of FGM plates, among which analytical and numerical methods based on certain simplified theories are frequently used. For example, Reddy et al. [1] examined the axisymmetric bending of functionally graded circular and annular plates by developing exact relationships between the solutions of the classical plate theory (CPT) and the first-order shear deformation plate theory (FSDT). Two refined displacement models, RSDT1 and RSDT2, were developed by Tounsi et al. [2] for a bending analysis of functionally graded sandwich plates. Exact analytical solutions directly based on the elasticity theory can only be derived for a relatively few problems, but they can serve as benchmarks for accessing the validity of various approximate plate theories or numerical methods. Cheng and Batra [3] used an asymptotic expansion method to analyze the isotropic FGM elliptic plate with clamped edges based on the three-dimensional (3D) elasticity theory. A 3D elasticity solution for an isotropic FGM rectangular plate with simply supported edges subject to transverse loading was developed by Kashtalyan [4]. Wang et al. [5] investigated the axisymmetric bending of transversely isotropic FGM circular plates subject to arbitrarily transverse loads. More works on FGM plate theories and their applications may be found in the review paper of Birman and Byrd [6].

It is noted that Mian and Spencer [7] developed an ingenious method to obtain a class of 3D solutions for isotropic FGM plates with traction-free surfaces, in which the material properties are assumed to vary arbitrarily with the thickness coordinate. Yang et al. [8] extended the above method to a transversely isotropic FGM annular plate with uniform loads applied on the top and bottom surfaces. Using the complex variables method, England [9] made a noticeable generalization of Mian and Spencer’s method [7] by including the effect of top-surface pressure, which satisfies the biharmonic equation or higher-order ones. Recently, Yang et al. [10] extended England’s method to the case of functionally graded plates with materials characterizing transverse isotropy; they obtained the elasticity solutions of an FGM rectangular plate with opposite edges simply supported and subject to a special family of biharmonic polynomial loads (totally 12 different types). To the authors’ knowledge, no analytical solution based on the 3D elasticity theory is found, which can be directly based on the elasticity theory can only be derived for a relatively few problems, but they can serve as benchmarks for accessing the validity of various approximate plate theories or numerical methods.

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2 Basic formulations

In the Cartesian coordinate system \((x, y, z)\), the equations of equilibrium in the absence of body forces can be written as

\[
\sigma_{ij,j} = 0, \quad (1)
\]

where the comma denotes differentiation with respect to the indicated variable.

The stress–displacement relations for transversely isotropic materials are expressed as [11]:

\[
\begin{align*}
\sigma_x &= c_{11}u_x + c_{12}v_y + c_{13}w_z, \\
\sigma_y &= c_{12}u_x + c_{11}v_y + c_{13}w_z, \\
\sigma_z &= c_{13}u_x + c_{13}v_y + c_{33}w_z, \\
\sigma_{xy} &= c_{44}(v_x + w_y), \\
\sigma_{xz} &= c_{66}(u_x + v_x), \\
\sigma_{yz} &= c_{66}(u_y + v_y), \\
\sigma_{zz} &= c_{66}(z),
\end{align*}
\]

(2)

where \(u, v \) and \(w\) are the displacement components, and \(c_{ij}\) with \(2c_{66} = c_{11} - c_{12}\) are the elastic constants. For FGMs, they are functions of \(z\), i.e., \(c_{ij}(z)\). If \(c_{11} = c_{33}, c_{12} = c_{13}, c_{44} = c_{66}\), the material becomes isotropic. The \(xy\) plane is an isotropic plane, coinciding with the mid-plane of the plate. The positive \(z\)-axis is upward and perpendicular to the mid-plane.

According to England [9], we seek the following solution of (1) and (2):

\[
\begin{align*}
u(x, y, z) &= \bar{w} + R_1 \Delta_x + R_0 \bar{w}_x + R_2 \nabla^2 \bar{w}_{xx} + R_3 \nabla^4 \bar{w}_{xxx} + R_4 \nabla^6 \bar{w}_{xxxx}, \\
w(x, y, z) &= \bar{w} + T_1 \Delta + T_2 \nabla^2 \bar{w} + T_3 \nabla^4 \bar{w} + T_4 \nabla^6 \bar{w},
\end{align*}
\]

(3)