Spheroidal inhomogeneity in a transversely isotropic piezoelectric medium

V. M. Levin, Th. Michelitsch, I. Sevostianov

Summary Piezoelectric material containing an inhomogeneity with different electroelastic properties is considered. The coupled electroelastic fields within the inclusion satisfy a system of integral equations solved in a closed form in the case of an ellipsoidal inclusion. The solution is utilized to find the concentration of the electroelastic fields around an inhomogeneity, and to derive the expression for the electric enthalpy of the electroelastic medium with an ellipsoidal inclusion that is relevant for various applications. Explicit closed-form expressions are found for the electroelastic fields within a spheroidal inclusion embedded in the transversely isotropic matrix. Results are specialized for a cylinder, a flat rigid disk and a crack. For a penny-shaped crack, the quantities entering the crack propagation criterion are found explicitly.

Key words Inclusion, piezoelectric material, electro-mechanical field

1 Introduction

Solutions for spheroidal inhomogeneities in a piezoelectric material are of key importance in connection with several problems. First, they constitute the basic building block for modeling the effective electroelastic properties of piezocomposites. Second, they yield concentration factors for the electroelastic fields near inclusions. Third, in the limiting case of a crack, the results yield the quantities that enter the crack propagation criterion, [1].

The problem of a spheroidal inclusion in a piezoelectric material has been considered by several authors. The classical approach of Eshelby was extended to the piezoelectric material in [2], but results were not derived in an explicit form. In [3, 4], electroelastic fields in the case of an ellipsoidal inclusion and in the limiting case of an elliptical crack were derived; however, the results were given in the form of integrals, containing Green’s function that was unknown at that time and could not, therefore, be readily used. General representation for an inhomogeneity of an arbitrary (not necessary ellipsoidal) shape was given in [5]. In [6, 7], Eshelby’s tensor was considered for an ellipsoidal inhomogeneity in a transversely isotropic and an orthotropic medium, correspondingly; however, similarly to [3, 4], these results were given in an integral, nonexplicit form due to the fact that Green’s function was not available in an explicit form at that time. Explicit expressions for components of Eshelby’s tensor for the spheroidal inhomogeneity were obtained in [8]. Similar results were derived by a different method in [9], where the limiting case of an infinite cylinder (a “fiber”) was also analyzed in detail.

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The present work constitutes further progress in studies of spheroidal inclusions in piezoelectric media. The new results obtained here can be outlined as follows. First, the electroelastic compliance tensors that describe the contribution of an inhomogeneity to the overall electroelastic response (and, thus, are of direct relevance for the effective electroelastic properties of a composite with multiple inclusions) are derived in the explicit form, in terms of elementary functions. These tensors constitute a generalization of the inclusion compliance tensors in the elasticity of materials with inclusions, [10, 11]. Second, general expressions for coefficients of electromechanical fields concentrations are derived. Third, in the case of a circular crack, the quantities that enter the crack propagation criterion proposed in [1] are explicitly calculated. Finally, the important asymptotic cases of strongly oblate and strongly prolate spheroids are analyzed in detail.

2 Electric and elastic fields in a medium with an inhomogeneity

In this section, we review some general results and modify them to a form suitable for the present work. We consider a homogeneous piezoelectric material under isothermal condition. The governing equations for such a material have the form

\[ \sigma_{ij} = C_{ijkl} e_{kl} - \epsilon_{ji} E_k, \quad D_j = e_{ijkl}^T e_{kl} + \eta_{ji} E_k, \]  \hspace{1cm} (1)

where \( \sigma, \epsilon \) are the stress and strain tensors, \( E, D \) are the electric field intensity and electric induction vectors, \( C \) is the tensor of elastic moduli, \( \eta \) is the tensor of dielectric permeabilities and \( e \) is the tensor of piezoelectric constants characterizing coupled electroelastic effects (the superscript \( T \) means the transpose).

Relations (1) can be written in the following short form:

\[ J = LF, \quad J = \begin{bmatrix} \sigma \\ D \end{bmatrix}, \quad L = \begin{bmatrix} C & e \\ e^T & -\eta \end{bmatrix}, \quad F = \begin{bmatrix} \epsilon \\ -E \end{bmatrix}, \]  \hspace{1cm} (2)

where the symbolic matrix \( L \) must be regarded as a linear operator, which transforms the tensor–vector pair \( [\sigma, D] \) into the pair \( [\epsilon, E] \).

The relations inverse to (1) have the form

\[ F = JL, \quad J = \begin{bmatrix} S \\ d^T \end{bmatrix}, \quad L = \begin{bmatrix} \kappa \\ -\kappa \end{bmatrix}, \]  \hspace{1cm} (3)

where

\[ S = (C + e\eta^{-1}e^T)^{-1}, \quad \kappa = (\eta + e^T C^{-1} e)^{-1}, \quad d = Se\eta^{-1} = C^{-1}e \kappa. \]

Let us consider now an infinite homogeneous piezoelectric body with the operator of electroelastic characteristics \( L^0 \), containing an inclusion with different operator of electroelastic constants \( L \) occupying a region \( \nu \). The strain \( \epsilon_{ij}(x) \) and electric intensity \( E_i(x) \) fields in an arbitrary point \( x \) of the medium with inhomogeneity satisfy the following system of integral equations, [12]:

\[ F(x) = F^0(x) + \int_{\nu} \mathcal{P}(x - x') L^1 F(x')dx', \]  \hspace{1cm} (4)

\[ \mathcal{P}(x) = \mathcal{P}(x)\mathcal{D}, \quad \mathcal{D} = \begin{bmatrix} \text{def} & 0 \\ 0 & \text{grad} \end{bmatrix}, \quad L^1 = L - L^0. \]

Here, \( F^0(x) \) stands for the external elastic and electric fields which would have taken place in the homogeneous matrix (without the inclusion) under the same boundary conditions. We assume in the present work that, in the absence of the inclusion, fields \( F^0(x) \) can be taken as constant at the length scale of \( \nu \). The kernel of this equation \( \mathcal{P}(x) \) is concentrated in the region \( \nu \) and expressed via the second derivatives of Green’s function \( \mathcal{G}(x) \) of the equilibrium equations of the coupled electroelasticity. This function satisfies the following equation: