On permanent long-wave distortions of a railway track due to the moving vehicle load

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Summary The paper presents a simple mathematical model which describes the evolution of the vertical profile of a railway track caused by the load arising from moving vehicles. Owing to the fact that the track ballast is not perfectly elastic, each passage of a train is accompanied by a residual deformation of the ballast. Being very small, these deformations, however, accumulate with time. It is shown that an initial imperfection in the vertical profile of the track will either grow or diminish after each passage. For given track and vehicle, the rate of growth is determined by the characteristic length of the imperfection and by the velocity of the vehicle.

Keywords Railway track, distortion, settlement

1 Introduction
The track ballast of railways is not perfectly elastic, and undergoes irreversible plastic deformation after each passage of a train. The residual deformation after one passage is rather small as compared to the elastic deformation of the track during the passage. However, after a large number of passages, the residual deformations accumulate. The question is whether the accumulation of the deformations will result in the growth of the track distortion or in the smoothing of the initial imperfection.

This paper presents a mathematical model of the formation of permanent long-wave distortions of a track resulting from small initial imperfections in the vertical profile. The problem of the differential settlement of a track with spatially varying stiffness was studied in [1, 2] where particular numerical solutions for given track geometry and stiffness were obtained. As distinguished from the studies [1, 2], we consider the problem of evolution of initial imperfections in the profile as a problem of stability of a straight profile to small disturbances to its geometry. The mechanical properties of the track are assumed to be homogeneous.

We are interested in long-wave distortions whose characteristic length is over 10 m. For vehicle velocities up to 250 km/h, this corresponds to frequencies below 7 Hz. We thus deal with low-frequency dynamics of the vehicle-track system. For this class of problems dynamics of the vehicle body plays an essential role, while the track’s inertia may be neglected, [3]. This makes it possible to consider a relatively simple mechanical model of the vehicle-track system omitting the elements which are unimportant for low-frequency dynamics.

The mathematical model of the vehicle-track interaction presented below allows us to calculate the residual deformation of the track after one passage if an initial imperfection in the vertical profile of the track is known. The evolution of an initial imperfection is determined by the relation between its shape and the residual deformations, which are successively added to the initial profile after each passage. It will be shown that an initial imperfection will either grow or diminish with time, which depends on the characteristic length of the imperfection and on the velocity of the vehicle.

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2  

Differential equation for the force

The mechanical model of the vehicle-track system used in the present study is shown in Fig. 1. In a fixed coordinate system \((x, y)\), a function \(f(x)\) represents the profile of the track (rail) before the passage. When a vehicle of mass \(m\) moves with constant horizontal velocity \(c\), the point of contact between the wheel and the rail (point \(A\)) moves along the curve \(w(x)\) which differs from \(f(x)\). The curve \(D\) represents the current profile of the track that changes during the passage. The current deflection of the track is assumed to be proportional to the force \(F\) acting on the rail at the point of contact with the wheel

\[
F = k_w(w - f) ,
\]

(1)

where \(k_w\) is a coefficient depending on the stiffness of both the rail and the railway bed. In (1) and in what follows, vertical forces are taken to be positive if the corresponding vectors are directed upward. We assume that the mechanical properties of the track and, specifically, its stiffness are uniform so that \(k_w\) in (1) is a constant. A function \(h(x)\) in Fig. 1 is the trajectory of point \(B\) and represents the trajectory of the vehicle body. If \(x\) is viewed as the current position of the vehicle, then \(f(x), w(x)\) and \(h(x)\) become functions of time \(t\) by the substitution \(x = ct + \text{const}\).

The vehicle body is connected with the wheelset through an elastic spring with stiffness \(k\) and a viscous element with viscosity \(\mu\). Therefore, the force \(F\) in (1) can also be written as

\[
F = k(h - w + C_1) + \mu \left(\frac{dh}{dt} - \frac{dw}{dt}\right) ,
\]

(2)

where \(C_1\) is a constant. The vertical motion of the mass is governed by the equation

\[
m \frac{d^2h}{dt^2} = -F - mg ,
\]

(3)

where \(g = 9.8 \text{ m/s}^2\) is acceleration due to gravity.

Our concern is how to find the force \(F(x)\) acting on the rail during the passage if an initial profile of the track, \(f(x)\), is given. Differentiating (1) with respect to time and substituting \(dw/dt\) and \(w\) from (1) into (2) allows us to eliminate the function \(w\) from (2). Then, differentiating (2) twice with respect to time and eliminating \(d^3h/dt^3\) and \(d^3h/dt^3\) with the use of (3), we obtain a third-order ordinary differential equation for \(F\) containing the function \(f\). In that equation, the differentiation with respect to time can be changed for the differentiation with respect to \(x\) with the use of the relation \(d/dt = c d/dx\), where \(c\) is the constant horizontal velocity of the vehicle. Finally, writing the function \(F(x)\) as the sum of the constant force \(F_0 = -mg\) and a varying part \(\bar{F}(x)\)

\[
F(x) = F_0 + \bar{F}(x) ,
\]

(4)

we arrive at the equation

\[
a_3 \frac{d^3\bar{F}}{dx^3} + a_2 \frac{d^2\bar{F}}{dx^2} + a_1 \frac{d\bar{F}}{dx} + a_0 \bar{F} = b_1 \frac{d^3f}{dx^3} + b_2 \frac{d^2f}{dx^2} ,
\]

(5)

![Fig. 1. Mechanical model of the vehicle-track system](image)