Hydrodynamic limit for $\nabla \phi$ interface model on a wall

Dedicated to Professor József Fritz at the occasion of his 60th birthday

Received: 10 January 2002 / Revised version: 18 August 2002 / Published online: 15 April 2003 – © Springer-Verlag 2003

Abstract. We consider random evolution of an interface on a hard wall under periodic boundary conditions. The dynamics are governed by a system of stochastic differential equations of Skorohod type, which is Langevin equation associated with a massless Hamiltonian added a strong repelling force for the interface to stay over the wall. We study its macroscopic behavior under a suitable large scale space-time limit and derive a nonlinear partial differential equation, which describes the mean curvature motion except for some anisotropy effects, with reflection at the wall. Such equation is characterized by an evolutionary variational inequality.

1. Introduction

The Ginzburg-Landau $\nabla \phi$ interface model is one of effective dynamical models for the interfaces created under the situation that phase coexistence and separation occur in the system. Assuming that complications like overhanging can be avoided for the interfaces and considering the system embedded in $d + 1$ dimensional space, the interface configurations are represented by their transversal deviations $\phi = \{\phi(x) \in \mathbb{R} ; x \in \mathcal{H}\}$ measured from a fixed $d$ dimensional flat hyperplane $\mathcal{H}$ located in $\mathbb{R}^{d+1}$. The hyperplane $\mathcal{H}$ is then discretized into the lattice $\mathbb{Z}^d$ and our object becomes $\phi : \mathbb{Z}^d \rightarrow \mathbb{R}$ with $\phi(x)$ the height of the interfaces at site $x \in \mathbb{Z}^d$.

To each configuration $\phi = \{\phi(x) ; x \in \mathbb{Z}^d\}$, interface energy is formally associated by an infinite sum

$$H(\phi) = \sum_{\langle x,y \rangle} V(\phi(x) - \phi(y)). \quad (1.1)$$

Here the sum is taken over all nearest neighbor bonds $b = \langle x, y \rangle : x, y \in \mathbb{Z}^d$, $|x - y| = 1$. We require the following three conditions on the potential $V$:
The role of these technical conditions on $V$, especially the condition (iii), is explained in [9].

The dynamics of Ginzburg-Landau type are naturally introduced corresponding to the energy $H(\phi)$. Namely, the evolutorial law of the interface $\phi_t = \{\phi_t(x); x \in \mathbb{Z}^d \}, t \geq 0$ can be defined by the Langevin equation

$$d\phi_t(x) = -\frac{\partial H}{\partial \phi(x)}(\phi_t) \, dt + \sqrt{2} d\phi_t(x), \quad x \in \mathbb{Z}^d,$$

(1.3)

where $\phi_t = \{\phi_t(x); x \in \mathbb{Z}^d \}$ is a family of independent standard one dimensional Brownian motions. Note that $H(\phi)$ itself is a formal infinite sum and has no apparent meaning, but its derivative $\partial H/\partial \phi(x)$ has a definite formula

$$\frac{\partial H}{\partial \phi(x)} = \sum_{y \in \mathbb{Z}^d; |x-y|=1} V(\phi(x) - \phi(y)).$$

Under suitable rescalings in space and time, the macroscopic behavior of the evolutorial of the interface $\phi_t$ defined at microscopic level can be studied from several aspects. In fact, the hydrodynamic limit (law of large numbers), equilibrium fluctuation (central limit theorem) and large deviations at static or dynamic levels were investigated by recent papers [9], [10], [5] and [7], respectively. The paper [6] reviews these results. The structure of all probability measures which are shift invariant and stationary under the time evolution for the corresponding $\nabla \phi$-field was also clarified by [9] under the conditions (1.2) on $V$, where the $\nabla \phi$-field is defined from $\phi$-field by $\nabla \phi(b) := \phi(x) - \phi(y)$ for all nearest neighbor bonds $b = (x,y)$ in $\mathbb{Z}^d$.

The aim of the present paper is to study the Ginzburg-Landau $\nabla \phi$ interface model on a hard wall. We shall establish its hydrodynamic behavior. The hyperplane $\mathcal{H}$ is now considered as a hard wall which strongly repels the interfaces. Accordingly, the height variables of the interface take only nonnegative values $\phi(x) \geq 0, x \in \mathbb{Z}^d$. In fact, we shall consider on the (lattice) torus $\Gamma_N := (\mathbb{Z}/N\mathbb{Z})^d \equiv \{1,2,\ldots,N\}^d$ rather than $\mathbb{Z}^d$. The dynamics of the interfaces are defined by the equation (1.3) added the effect of a hard core potential located at $\mathcal{H}$, which is mathematically formulated as a local time at $\phi(x) = 0$ for each site $x \in \Gamma_N$, see the equation (2.3) with (2.4) below.

The main result is stated in Theorem 2.1 in Section 2. The limiting hydrodynamic equation is described by an evolutionary variational inequality (EVI). In Section 3, the existence and uniqueness for the EVI are established. One of the important tools for our study will be comparison theorems for the SDEs, which are available as a consequence of the convexity of the potential $V$; see Section 4. We replace the singular drift terms given by the local times in the equation (2.3) with nonsingular drifts. This procedure called penalization introduces the lower and upper bounds for the solution of (2.3). The hydrodynamic limits are established in Sections 5 and