Abstract. We continue the investigation of noncommutative cumulants. In this paper various characterizations of generalized Gaussian random variables are proved.

1. Introduction

Generalized Gaussian random variables and Brownian motions have a long history in noncommutative probability theory and noncommutative central limit theorems. For a systematic study see [GM02]. In this paper we consider generalized Gaussian random variables from the point of view of combinatorial cumulant theory as developed in our paper [Leh02], to which we refer as part I. Our aim is to prove characterizations of Gaussian random variables, as found in [KLR73] and [Bry95] for classical Gaussian distributions. There are essentially two kinds of characterizations. The proofs of the simpler ones like spherical symmetry, Bernstein’s and Lukacs’ theorems can be immediately transferred to the noncommutative case, while other theorems, notably including Cramer’s and Marcinkiewicz’ theorems, do not hold in general.

Throughout this paper we will consider a fixed noncommutative probability space $(\mathcal{A}, \varphi)$ and an exchangeability system $\mathcal{E} = (\mathcal{U}, \tilde{\varphi}, \mathcal{J})$ for $(\mathcal{A}, \varphi)$ as defined in part I. The interchangeable images of $\mathcal{A}$ in $\mathcal{U}$ will as usual be denoted by $(\mathcal{A}_i)_{i \geq 0}$, and we shall identify $\mathcal{A}$ with $\mathcal{A}_0$. We shall moreover assume that $\mathcal{A}$ is a $*$-algebra and that all considered random variables are selfadjoint.

Definition 1.1. We say that two random variables $X$ and $Y \in \mathcal{A}$ have the same distribution given $\mathcal{E}$, if for any word $W = W_1W_2 \cdots W_n$ with $W_i \in [X] \cup \bigcup_{i \geq 1} \mathcal{A}_i$ the expectation $\tilde{\varphi}(W)$ does not change if we replace each occurrence of $X$ by $Y$. We call $X$ and $Y$ $\mathcal{E}$-i.i.d. if in addition they are $\mathcal{E}$-independent. Similarly a sequence $(X_i)_{i \in \mathbb{N}} \subseteq \mathcal{A}$ of $\mathcal{E}$-independent random variables is called $\mathcal{E}$-i.i.d. if for any word $W = W_1W_2 \cdots W_n$ with $W_i \in [X_i : i \in \mathbb{N}] \cup \bigcup_{i \geq 1} \mathcal{A}_i$ the expectation $\tilde{\varphi}(W)$
does not change if we apply a permutation \( \sigma \in \mathfrak{S}_\infty \) to the indices of \( X_i \), i.e., if we replace each occurrence of \( X_i \) by \( X_{\sigma(i)} \).

We will need the following weak variant of pyramidal independence (cf. Definition I.3.10).

**Definition 1.2.** Let \( X_i \) be an interchangeable sequence of (centered) random variables, that is, for every permutation \( \pi \in \mathfrak{S}_\infty \) and every choice of indices \( i_1, i_2, \ldots, i_n \) the expectation does not change under permutations:

\[
\phi(X_{\pi(i_1)}X_{\pi(i_2)} \cdots X_{\pi(i_n)}) = \phi(X_{i_1}X_{i_2} \cdots X_{i_n}).
\]

We say that the singleton condition holds if

\[
\phi(X_{i_1}X_{i_2} \cdots X_{i_n}) = 0
\]

whenever one of the \( X_i \)’s occurs exactly once.

Let us start this section by quoting a general noncommutative central limit theorem.

**Theorem 1.3 ([BS96]).** Let \((A, \phi)\) be a noncommutative probability space, and \( X_i = X_i^* \in A \) be a sequence of exchangeable random variables. For a partition \( \nu \) denote

\[
\phi(\nu) = \phi(X_{i_1}X_{i_2} \cdots X_{i_n})
\]

where \( (i_1, i_2, \ldots, i_n) \) is any multiindex with kernel \( \nu \). Assume that \( \phi(X_i) = 0 \), \( \phi(X_i^2) = 1 \) and moreover that the singleton condition of Definition 1.2 holds. Then the sequence \( S_N = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} X_i \) has limit distribution

\[
\lim_{N \to \infty} \phi(S_{N}^{(2n+1)}) = 0 \quad \lim_{N \to \infty} \phi(S_{N}^{(2n)}) = \sum_{\nu \in \mathfrak{P}_{\text{pair}}} \phi(\nu)
\]

Interchangeable sequences generate interchangeable algebras and give rise to exchangeability systems. In view of the preceding noncommutative central limit theorem we define Gaussian families as follows (see also [GM02]).

**Definition 1.4.** An interchangeable family \( (X_i) \) of random variables is called (centered) Gaussian if all cumulants which correspond to non-pair partitions vanish. In other words, there is a function on pair partitions \( \nu : \mathfrak{P}_{\text{pair}}^{(2)} \to \mathbb{C} \) such that for all \( h : [n] \to I \)

\[
\phi(X_{h(1)}X_{h(2)} \cdots X_{h(n)}) = \sum_{\pi \in \mathfrak{P}_{\text{pair}}^{(2)} \pi \leq \ker h} \nu(\pi)
\]

In particular, odd moments vanish and the singleton condition holds.

Noncommutative (i.e. operator valued) Khinchin inequalities are available for Gaussian families, see [Buc01].

In the following all random variables are assumed self-adjoint and the state \( \bar{\phi} \) is assumed to be faithful. This is needed for the following crucial lemma to be valid.