A functional central limit theorem for diffusions on periodic submanifolds of $\mathbb{R}^N$

Abstract. We prove a functional central limit theorem for diffusions on periodic submanifolds of $\mathbb{R}^N$. The proof is an adaptation of a method presented in [BenLioPap] and [Bha] for proving functional central limit theorems for diffusions with periodic drift vector-fields. We then apply the central limit theorem in order to obtain a recurrence and a transience criterion for periodic diffusions. Other fields of applications could be heat-kernel estimates, similar to the ones obtained in [Lot].

1. Introduction

Let $M$ be a closed, connected sub-manifold of $\mathbb{R}^N$. We assume that there exists a lattice $\Lambda \subset \mathbb{R}^N$ such that the translation by elements of $\Lambda$ maps $M$ into $M$. Furthermore, we assume that $M/\Lambda$ is compact and has an orientation. As a sub-manifold of the Riemannian manifold $\mathbb{R}^N$, $M$ carries a natural Riemannian metric. The associated Riemannian volume-measure will be denoted with $v_0$. Let $L$ be a periodic, elliptic differential operator of second order defined on $C^2(M)$. The operator $L$ generates a diffusion-process $X$ on $M$ (see [Hsu] p.24). For every function $g \in C^\infty(M)$

$$M_t^g := g(X_t) - g(X_0) - \int_0^t Lg(X_s)ds$$

is a local martingale with respect to a suitable filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq0})$. As in [Hsu] we assume the filtration to be complete and right continuous. Since the generator $L$ is periodic, the diffusion process $X$ on $M$ can be naturally identified with a diffusion $X^{\Lambda}$ on $M/\Lambda$. Since $L$ is elliptic, it generates a strongly continuous contraction semigroup $t \mapsto e^{tL}$ on $C(M)$ and it has positive fundamental solutions $p : \mathbb{R}^+ \times M \times M \to \mathbb{R}^+$ with respect to $v_0$. In local coordinates $L$ has the following expression

B. Franke: Ruhr Universität Bochum, 44780 Bochum, Germany.

e-mail: Brice.Franke@ruhr-uni-bochum.de

Mathematics Subject Classification (2000): 35B27, 60F05, 58J65

Key words or phrases: Functional central limit theorem – Homogenization – Asymptotic analysis – Periodic diffusion – Periodic manifold – Recurrence – Transience

The author wants to express his gratitude toward the National Cheng Kung University in Tainan (Taiwan) for its kind hospitality.
A functional central limit theorem for diffusions on periodic submanifolds of $\mathbb{R}^N$

\[
L = \frac{1}{2} \sum_{i,j} a^{ij} \partial_i \partial_j + \sum_{i=1}^d b^i \partial_i
\]

with smooth coefficients $a^{ij}, b^i$. Furthermore, one defines for $h, g \in C^\infty(M)$

\[
\Gamma(h, g) = L(hg) - hLg - gLh.
\]

In local coordinates $\Gamma$ takes the following form

\[
\Gamma(h, g) = \sum_{i,j} a^{ij}(\partial_i h)(\partial_j g).
\]

Since $M/\Lambda$ is compact, there exists an invariant probability measure $\mu$ for $X^\Lambda$ on $M/\Lambda$, and two constants $C, \lambda > 0$ such that for all periodic $g \in L^\infty(M, v_\Lambda)$ with $\int_M g d\mu = 0$ one has

\[
\|e^{tL}g\|_\infty \leq C\|g\|_\infty e^{-\lambda t}.
\]

Therefore, the resolvent in zero of $L$ is defined on the orthogonal complement of the constant functions in $L^2_p(M, \mu)$. This implies that for all $g \in L^2_p(M, \mu)$ with $\int_M g d\mu = 0$ the Poisson-problem $L\psi = g$ has a solution in $\psi \in L^2_p(M, \mu)$. By elliptic regularity $\psi$ is in $C^\infty(M)$ if $g \in C^\infty(M)$. For $1 \leq \alpha \leq N$ the restriction of the coordinate functions $k^\alpha : \mathbb{R}^N \to \mathbb{R}; x \mapsto x^\alpha$ to $M$ will be denoted by $f^\alpha$. We note that $Lf^\alpha \in L^2_p(M, \mu) \cap C^\infty(M)$. Therefore, the Poisson-problem

\[
L\psi^\alpha = Lf^\alpha - \int_{M/\Lambda} Lf^\alpha d\mu
\]

has a solution in $L^2_p(M, \mu) \cap C^\infty(M)$. We denote by $\overline{L}^\alpha$ the vector in $\mathbb{R}^N$ with components $\overline{L}f^\alpha := \int_{M/\Lambda} Lf^\alpha d\mu$ for $1 \leq \alpha \leq N$. Since $M$ is a sub-manifold of $\mathbb{R}^N$ the diffusion $X$ can also be interpreted as a semi-martingale in $\mathbb{R}^N$ (see [Hsu] p.21). Therefore, $X$ can be viewed as a random variable taking its values in the space of càdlàg functions $D_{\mathbb{R}^N}([0, \infty])$ with the usual Skorohod topology (see [EthKur] p.118). We want to show that the distributions of the rescaled semi-martingales

\[
X_t^{(n)} := n^{-1/2}(X_{nt} - X_0 - nt\overline{L})
\]

converge in the weak sense to the distribution of a Gaussian martingale with independent increments. For a given positive semi-definite, symmetric $N \times N$-matrix $\Sigma$, there exists a unique Gaussian $\mathcal{F}_t$-martingale $W^\Sigma$ on $\mathbb{R}^N$ starting in zero with covariation process $\langle W^\Sigma, W^\Sigma \rangle_t = t \Sigma$ for all $t \geq 0$ and $\mathbb{P}$-a.s. (see [EthKur] p.338). By Levy’s characterization theorem the projections of $W^\Sigma$ onto one dimensional subspaces of $\mathbb{R}^N$ are multiples of Brownian motions (see [RevYor] p.141).