Abstract. In this paper, we construct one Yang-Mills measure on an orientable compact surface for each isomorphism class of principal bundles with compact connected structure group over this surface. For this, we refine the discretization procedure used in a previous construction [9] and define a discrete theory on a new configuration space which is essentially a covering of the usual one. We prove that the measures corresponding to different isomorphism classes of bundles or to different total areas of the base space are mutually singular. We give also a combinatorial computation of the partition functions which relies on the formalism of fat graphs.

Introduction

The Yang-Mills measure is the law of a group-valued random process indexed by a family of paths on some manifold. It is usually thought of as the random holonomy induced by a probability measure on the space of connections on a principal bundle over this manifold. We consider in this paper the case where the base manifold is an oriented compact surface and the structure group is a compact connected Lie group. In this case, the measure has been studied at various levels of rigor by several authors. In particular, the origin of its mathematical study is a paper by the physicist A. Migdal [13]. Other important contributions are those of B. Driver [3, 2] and, with different motivations, E. Witten [20]. The first rigorous construction has been given by A. Sengupta [16], by conditioning an infinite-dimensional noise. A second construction has been given by the author in [9], where the random holonomy process is built by passing discrete approximations to the limit. This leads essentially to the same object, but in a way that gives a different and sometimes better grip on it (see for example [11]).

A basic feature of the whole Yang-Mills theory is its invariance under gauge transformations. Accordingly, the Yang-Mills measure on a specific principal bundle depends on this bundle only up to isomorphism. On the other hand, it can and indeed should depend on this isomorphism class: connections on a principal bundle carry a lot of topological information about the bundle itself, for instance through
characteristic classes. A Sengupta’s construction produces one measure for each isomorphism class of principal bundle but the author’s construction in [9] doesn’t. We shall see that the measure constructed there corresponds to an average over all possible isomorphism classes. The aim of this paper is to fill this gap. More precisely, we propose a construction of the Yang-Mills measure by using a discrete approximation and passing it to the continuous limit in a way that keeps track of the topology of the bundle.

This may sound paradoxical for the following reason. Discrete approximations of the Yang-Mills measures are usually built by first considering graphs on the base manifold and restricting the bundle to these one-dimensional complexes. But, as long as the structure group is connected, the restricted bundle is always trivial and any topological information about the full bundle is lost in this operation. This is why the construction of [9] produces a single probability measure, which is associated to no particular isomorphism class of fibre bundles, but rather to a random bundle for some natural probability measure on the set of isomorphism classes. In the case of an Abelian structure group, this was to some extent already understood, because in this case it is easier to compare A. Sengupta’s construction and ours (see the informal remarks in [9, Sections 1.9.3 and 3.2]).

The first section of this paper is devoted to the construction of a new discrete theory, namely a finer way of discretizing the differential geometric objects involved than the naive one commonly used in discrete gauge theory. This new discretization is fine enough to capture the topology of the bundle. In fact, what we do is replacing the usual configuration space of discrete gauge theory by a singular covering of it, which happens to be finite when the structure group is semi-simple. In all cases, the singular set is negligible and plays no role at the level of measure theory. A partial study of the topological structure of this singular covering is presented in the last section.

In the second and third sections, we construct the Yang-Mills measures associated to a specific isomorphism class of principal bundles. For this, we consider first semi-simple structure groups, because in this case, as explained above, the new configuration space is a singular finite covering of the usual one. Then, although the method used to construct the discrete measures does not extend to general structure groups, the formula derived in the semi-simple case make sense without the semi-simplicity assumption. This allows us to construct the discrete measures for general compact connected structure groups. In the case of semi-simple and Abelian groups, we check that our construction is consistent with the previous work of A. Sengupta and the remarks of [9] mentioned above. Finally, in the third section, we pass these discrete measures to the continuous limit, following step by step the construction presented in the second chapter of [9]. However, in counterpart for the fact that it carries more geometric informations, the new discretization does not produce a nice projective family of probability spaces as the usual procedure does. It is thus necessary to go back to the usual configuration spaces before passing to the limit. As a consequence, we construct probability measures on the same space as in [9], not on some covering of it.