Invariant random graphs with iid degrees in a general geography

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Abstract Let $D$ be a non-negative integer-valued random variable and let $G = (V, E)$ be an infinite transitive finite-degree graph. Continuing the work of Deijfen and Meester (Adv Appl Probab 38:287–298) and Deijfen and Jonasson (Electron Comm Probab 11:336–346), we seek an $\text{Aut}(G)$-invariant random graph model with $V$ as vertex set, iid degrees distributed as $D$ and finite mean connections (i.e., the sum of the edge lengths in the graph metric of $G$ of a given vertex has finite expectation). It is shown that if $G$ has either polynomial growth or rapid growth, then such a random graph model exists if and only if $\mathbb{E}[D R(D)] < \infty$. Here $R(n)$ is the smallest possible radius of a combinatorial ball containing more than $n$ vertices. With rapid growth we mean that the number of vertices in a ball of radius $n$ is of at least order $\exp(nc)$ for some $c > 0$. All known transitive graphs have either polynomial or rapid growth. It is believed that no other growth rates are possible. When $G$ has rapid growth, the result holds also when the degrees form an arbitrary invariant process. A counter-example shows that this is not the case when $G$ grows polynomially. For this case, we provide other, quite sharp, conditions under which the stronger statement does and does not hold respectively. Our work simplifies and generalizes the results for $G = \mathbb{Z}$ in [4] and proves, e.g., that with $G = \mathbb{Z}^d$, there exists an invariant model with finite mean connections if and only if $\mathbb{E}[D^{(d+1)/d}] < \infty$. When $G$ has exponential growth, e.g., when $G$ is a regular tree, the condition becomes $\mathbb{E}[D \log D] < \infty$. 

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1 Introduction

In recent years there has been an increasing interest in the use of random graph models as models for different complex structures. For such applications the original Erdős–Rényi model will not do. One reason for this is that the degree distribution of these structures is widely different from what one gets from the Erdős–Rényi model. Therefore it has been a natural step to construct models, where the degrees of different vertices are iid random variables with a given desired distribution \( F \). A handful of models of this type have been proposed by different authors, see \([4,5]\) and the references therein.

A second reason for the need for new random graph models is that many of the networks one wants to model, exhibit a notion of geography; the vertices have a well-defined position in space. Therefore it is natural to ask for models which are, in addition to the above, also geographically invariant in some proper sense. This problem was introduced by Deijfen and Meester \([5]\), who constructed an invariant model on \( \mathbb{Z} \) (i.e., a random graph model on the vertices of \( \mathbb{Z} \) whose edge configuration is invariant under the automorphisms of \( \mathbb{Z} \)). Their model leads to well-defined graphs, provided that \( F \) has finite mean. However, the expected edge lengths, and hence the expected total edge length of a vertex, turn out to be infinite for any \( F \). This leads to the question if one can construct models where this is not the case and, if so, what is a necessary and sufficient condition on \( F \) for this to be possible. A first answer came in \([4]\) where an invariant model on \( \mathbb{Z} \) with finite expected total edge length of a vertex, was constructed under the condition that \( F \) has finite second moment. It is easily seen that finite second moment is also necessary.

The present paper is a natural continuation of \([4]\); we seek to extend the results from there to other geographies. We will do this in the most general sense possible. Let \( G = (V, E) \) be an infinite transitive finite-degree graph with either polynomial or rapid (for definition, see the next section) growth. (In fact, it is believed, but still not confirmed, that no other growth rates are possible.) We will prove the existence of an \( \text{Aut}(G) \)-invariant model on \( G \), with finite expected total edge length per vertex, when \( \mathbb{E}[DR(D)] < \infty \). Here \( R(n), n = 1, 2, \ldots \), is the radius of the smallest possible combinatorial ball with more than \( n \) vertices and \( D \) is distributed according to \( F \).

In the polynomial growth case, our model will be based on a discrete version of a “stable marriage of Poisson and Lebesgue” of Hoffman et al. \([7]\). For \( G = \mathbb{Z} \) this model is similar in spirit to the one in \([4]\), but it turns out to be more amenable to generalization. For \( G = \mathbb{Z}^d, d \geq 1 \), \( R(n) \) is of order \( n^{1/d} \) so the condition \( \mathbb{E}[DR(D)] < \infty \) becomes \( \mathbb{E}[D^{(d+1)/d}] < \infty \). When \( G \) is a regular tree of degree at least 3, we get \( \mathbb{E}[D \log D] < \infty \). The latter condition also applies to some more exotic geographies.