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**Eigenvalue distributions of random unitary matrices**

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**Abstract.** Let $U$ be an $n \times n$ random matrix chosen from Haar measure on the unitary group. For a fixed arc of the unit circle, let $X$ be the number of eigenvalues of $M$ which lie in the specified arc. We study this random variable as the dimension $n$ grows, using the connection between Toeplitz matrices and random unitary matrices, and show that $(X - E[X])/(\operatorname{Var}(X))^{1/2}$ is asymptotically normally distributed. In addition, we show that for several fixed arcs $I_1, \ldots, I_m$, the corresponding random variables are jointly normal in the large $n$ limit.

1. Introduction

Let $U$ be a matrix chosen according to Haar measure from the group $U_n$ of $n \times n$ unitary matrices. All of the eigenvalues of such matrices lie on the complex unit circle, and a great deal of work has been done investigating the distribution of eigenvalues. Fix a finite number of intervals on the unit circle $I_1 = (e^{i\alpha_1}, e^{i\beta_1}), \ldots, I_m = (e^{i\alpha_m}, e^{i\beta_m})$. We study the number of eigenvalues of a random unitary matrix lying in these intervals. Define random variables $X_{n1}, \ldots, X_{nm}$ by

$$X_{nk}(U) = \sum_{j=1}^{n} 1_{\{\lambda_j \in I_k\}}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of $U$. For large $n$, the mean and variance asymptotics of $X_{nk}$ are known to be

$$E_n(X_{nk}) = \frac{n(\beta_k - \alpha_k)}{2\pi}, \quad (1)$$

and

$$\operatorname{Var}(X_{nk}) = \frac{1}{\pi^2} \left( \log n + 1 + \gamma + \log \left| 2 \sin \left( \frac{\beta_k - \alpha_k}{2} \right) \right| \right) + o(1). \quad (2)$$

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(The expectation follows from the symmetry of the unitary group; the complete variance asymptotic expansion was calculated by Eric Rains in [22].)

Now let \( Z = (Z_1, \ldots, Z_m) \) be jointly distributed normal random variables, where \( E[Z_k] = 0, \text{Var}(Z_k) = 1, \) and

\[
\text{Cov}(Z_j, Z_k) = \begin{cases} 
0 & \text{if } \alpha_j, \alpha_k, \beta_j, \text{ and } \beta_k \text{ are distinct} \\
1/2 & \text{if } \alpha_j = \alpha_k \text{ or if } \beta_j = \beta_k \text{ (but not both)} \\
-1/2 & \text{if } \alpha_j = \beta_k \text{ or if } \beta_j = \alpha_k \text{ (but not both)} \\
1 & \text{if } \alpha_j = \alpha_k \text{ and } \beta_j = \beta_k \\
-1 & \text{if } \alpha_j = \beta_k \text{ and } \beta_j = \alpha_k.
\end{cases}
\]

(3)

Also, for \( k = 1, \ldots, m, \) define

\[
Y_{nk} = \frac{X_{nk} - E[X_{nk}]}{\frac{1}{\pi} (\log n)^{1/2}}.
\]

The main result of this paper is the following:

**Theorem 1.** As \( n \to \infty, (Y_{n1}, \ldots, Y_{nm}) \) converges in distribution to \((Z_1, \ldots, Z_m)\).

As \( n \) goes to infinity, the limit of the \( n \)-dimensional eigenvalue distribution is a random point field which is determined by its correlation functions. In [9], Costin and Lebowitz proved a single interval case of Theorem 1 for this random point field, by studying the behavior of the cumulants. While their calculations were done for a specific kernel, they noted on page 71 of their paper that the proof relies only on a few properties of the underlying kernel and that their result in fact covers a much larger class of matrix models. The general version of their result, which is proved in [28] (and can also be found in [27]), includes the random unitary matrices. The multiple interval case of Theorem 1 is also stated in [9]; however the paper does not include a proof, and the extension to multiple intervals is not entirely straightforward.

The present work is drawn from [34]. The proof given here is based on a well-known connection between probability on \( U_n \) and determinants of \( n \times n \) Toeplitz matrices. The main technical result is Theorem 2 (section 4), a product theorem for Toeplitz matrices and operators. The theorem generalizes a Toeplitz matrix result due to Basor [1].

Two recent works have alternative approaches to the multiple interval case of Theorem 1. Soshnikov [28] has discussed the multiple interval case using the approach of [9], and Diaconis and Evans [10] have found a proof of Theorem 1 using methods introduced in [11] studying traces of powers of random unitary matrices. Related results for the case when the interval’s length decreases as the dimension of the matrix increases can be found in [10] and in [29]. Also closely related are the results of Keating and Snaith [20] for the characteristic polynomial of a random unitary matrix and the paper by Hughes, Keating, and O’Connell [15]; in particular [15] provides an explanation of the unusual covariance (3).

The paper is organized as follows. Some background on Toeplitz matrices and asymptotics is given in the next section. The third section contains some basic facts