Almost sure stability of linear stochastic differential equations with jumps

C. W. Li* · Z. Dong · R. Situ

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Abstract. Under the nondegenerate condition as in the diffusion case, see [14, 21, 6], the linear stochastic jump-diffusion process projected on the unit sphere is a strong Feller process and has a unique invariant measure which is also ergodic using the relation between the transition probabilities of jump-diffusions and the corresponding diffusions due to [22]. The unique deterministic Lyapunov exponent can be represented by the Furstenberg-Khas’minskii formula as an integral over the sphere or the projective space with respect to the ergodic invariant measure so that the almost sure asymptotic stability of linear stochastic systems with jumps depends on its sign. The critical case of zero Lyapunov exponent is discussed and a large deviations result for asymptotically stable systems is further investigated. Several examples are treated for illustration.

1. Introduction

Gaussian processes have been used to model many physical systems for quite a long time. However it is becoming an increasing requirement in modeling to use Poisson processes, which exhibit jump-like behavior. Such processes are being identified in the physical environment, in engineering systems and in financial sectors. For example, random telegraph processes or any finite-state continuous-time Markov chains with infinitesimal transition probability matrices can be modeled by stochastic differential equations driven by Poisson processes, see Example 4.4 in section 4.

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C. W. Li: Department of Mathematics, City University of Hong Kong, 83 Tat Chee Avenue, Kowloon, Hong Kong. E-mail: macwli@cityu.edu.hk (corresponding author)

Z. Dong: Institute of Applied Mathematics, Academy of Mathematics and Systems Sciences, Academia Sinica, Beijing 100080, P. R. China. Research supported in part by NSFC grant 19801003.

R. Situ: Department of Mathematics, Zhongshan University, Guangzhou 510275, P. R. China

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below. We will adopt the jump-diffusion model to describe the continuous and jump noises simultaneously.

For the linear random dynamical systems in \( \mathbb{R}^d \), there exists at most \( d \) Lyapunov exponents and the corresponding invariant subspaces as assured by Oseledec’s Multiplicative Ergodic Theorem in [20]. If the linear random dynamical system has an infinitesimal generator, then the top Lyapunov exponents can be represented by the Frustenberg-Khas’minskii formula as an integral over the unit sphere or the projective space with respect to an invariant measure, see Arnold [1, Chapter 6]. The linear diffusion case is classical and can be found in Khas’minskii [14, 15], Pinsky and et al [10, 21], Arnold and et al [1–6], while the pure jump case is treated by Li and Blankenship [19]. The invariant measures for diffusion processes are studied by Arnold and Kliemann [3, 5] and Kliemann [17]. The moment stability for diffusions is investigated by Arnold [2], Arnold, Oeljeklaus and Pardoux [6], Baxendale [7], Khas’minskii and Moshchuk [16]. Large deviations have been studied by Stroock [23], Arnold and Kliemann [4], Baxendale and et al [7, 8].

In this article we would like to extend the results for diffusions to jump-diffusions. As there is insufficient knowledge about the existence and uniqueness of solutions of integro-differential equations, it is much difficult to determine invariant measures for jump-diffusion processes. However due to Skorokhod [22], the gap between jump-diffusions and the corresponding diffusions is bridged.

Let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})\) be the underlying filtered complete probability space. Consider the following system of the linear stochastic differential equation with jumps

\[
dx(t) = Ax(t)dt + \sum_{i=1}^{m} B_i x(t) \circ dW_i(t) + \int_{Z} C(z) x(t-z) N(dt, dz) \tag{1.1}
\]

with initial condition \( x(0) = x_0 \neq 0 \) in \( \mathbb{R}^d \). The symbol \( \circ \) denotes the Stratonovich calculus, \( W = (W_1, \ldots, W_m) \) is a standard \( m \)-dimensional Brownian motion and \( N(dt, dz) \) is a Poisson point process on \( \mathbb{R}_+ \times Z \) with simultaneous jumps of probability zero with \( \lambda(dz) \), the deterministic finite characteristic measure on a measurable space \( Z \) such that \( \tilde{N}(dt, dz) = N(dt, dz) - \lambda(dz)dt \) is a \( \mathcal{F}_t \)-martingale measure. We also assume \( W \) and \( N \) are independent; \( A, B_k = (b_{ij}^k), k = 1, \ldots, m, C(z) \) are \( d \times d \) matrices such that \( C \in L^{d+1}(Z) \). Define

\[
\tilde{A} = A + \frac{1}{2} \sum_{k=1}^{m} B_k^2, \quad \tilde{C}(z) = I + C(z).
\]

Further we may assume the nondegenerate condition on the jump part:

\[
|\tilde{C}(z)x| \geq \eta(z)|x|, \quad x \in \mathbb{R}^d, \tag{1.2}
\]

for some constant \( \eta(z) > 0, \lambda \)-a.e. on \( Z \). The condition (1.2) is equivalent to the full rank of \( \tilde{C}(z) \), \( \lambda \)-a.e. on \( Z \). Define an operator \( \bullet \) on \( \mathbb{S}^{d-1} \) by \( \tilde{C}(z) \bullet x = \tilde{C}(z)x/|\tilde{C}(z)x|, x \in \mathbb{R}^d \). Let \( f \in C^2(\mathbb{R}^d) \). The formal integro-differential operator corresponding to (1.1) is