An Approximation Algorithm for Circular Arc Colouring¹

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Abstract. We consider the problem of colouring a family of $n$ arcs of a circle. This NP-complete problem, which occurs in routing and network design problems, is modelled as a 0-1 integer multicommodity flow problem. We present an algorithm that routes the commodities in the network by augmenting the network with some extra edges which correspond to extra colours. The algorithm, which relies on probabilistic techniques such as randomized rounding and path selection, is a randomized approximation algorithm which has an asymptotic performance ratio of $1 + 1/e$ (approximately 1.37) except when the minimum number of colours required is very small ($O(\ln n)$). This is an improvement over the best previously known result [7], which is a deterministic approximation algorithm with a performance ratio of $3/2$. The substantial improvement is valuable, for instance in wavelength allocation strategies in communication networks where bandwidth is a precious resource.

Key Words. Circular arc graph, Colouring, Approximation algorithms, Randomized rounding.

1. Introduction. The circular arc colouring problem is the problem of finding a minimal colouring of a set of arcs of a circle such that no two overlapping arcs share a colour. A circular arc graph is a representation of a set of arcs where the vertices correspond to the arcs and each edge represents an overlap. A valid vertex colouring of a circular arc graph corresponds to a valid colouring of the corresponding set of arcs.

Applications include problems in network design and scheduling. Many resource allocation problems in ring-structured networks have a similar flavour. Rings are a common architecture in communication networks: they combine the advantages of biconnectivity with a low cost and simple design. A common example of ring-structured networks are SONET rings [2]. The task of allocating wavelengths to communication paths in a ring network is analogous to colouring a set of arcs. The number of colours required represents the bandwidth requirement in the communication network.

There have been several investigations of the circular arc colouring problem [5], [7], [12], [13], [15]. The problem was shown to be NP-complete by Garey et al. in [5]. Tucker [15] reduced the problem to an integral multicommodity flow problem. For the special case of the proper circular arc colouring problem, polynomial time algorithms exist, such as an $O(n^{1.5})$ algorithm due to Shih and Hsu [12]. A set of circular arcs is

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proper if no arc is contained in another. For the general case, Tucker [15] gave a simple approximation algorithm with an approximation ratio of 2. This algorithm divides the arcs into two sets, one containing all the arcs passing over a particular point \( P \) on the circle, and the other containing the rest. Colouring the two sets of arcs is equivalent to solving two instances of the problem of colouring an interval graph, for which optimal algorithms exist [6]. It is not difficult to see that the number of colours needed to colour either set of arcs is no more than the minimal number of colours required for the original problem, yielding a performance ratio of 2.

An approximation algorithm with a performance ratio of \( \frac{5}{3} \) is due to Shih and Hsu [13]. Karapetian [7] presents an algorithm to colour a set \( S \) of arcs with no more than \( \lceil \frac{3}{2} \psi(S) \rceil \) colours, where \( \psi(S) \) is the size of the largest subset of \( S \) in which any two arcs intersect. The algorithm achieves an approximation ratio of \( \frac{1}{2} \), since \( \psi(S) \) is a lower bound on the number of colours required to colour \( S \). Approximation algorithms are relevant in applications like network design, where even a small improvement in performance ratio translates into a saving of precious bandwidth.

In this paper we present a randomized approximation algorithm that achieves a performance ratio of \( 1 + \frac{1}{e} + o(1) \) for instances where the minimum number of colours required is not very small. More precisely, the minimum number of colours required must be \( \omega(\ln n) \), where \( n \) is the number of arcs to be coloured. We begin by modelling the problem as an integral multicommodity flow problem, using a reduction due to Tucker [15]. We set up the multicommodity flow problem as an integer program, and use LP relaxation to obtain an optimal non-integral solution. Next, we randomly select integer values for fractional quantities to get an integral solution close to the optimal non-integral solution. We show that with high probability, the randomly generated integral solution will not be too far from the optimal non-integral solution in terms of optimality. This technique of using LP relaxation and random generation to obtain provably good solutions to integral optimization problems is called randomized rounding, and is due to Raghavan [9], [10].

1.1. Paper Outline. The rest of the paper is organized as follows. Section 2 deals with the circular arc graph colouring problem. The problem is modelled as a multicommodity flow problem in Sections 2.1 and 2.2. Section 3 details a solution to the flow problem, and the performance of our algorithm is analysed in Section 4. Section 5 summarizes our results and conclusions, and the Appendix contains the proof of a useful mathematical result.

2. Arc Colouring and Multicommodity Flows. We are given a family \( F \) of arcs, such as the one shown in Figure 1. Our objective is to find a colouring of \( F \) that uses the smallest possible number of colours.

An overlap set is the set of all arcs in \( F \) that contain some particular point \( p \) on the circle. We refer to the size of the largest overlap set as the width of \( F \). Let \( p_0, p_1, \ldots, p_{n-1} \) be the \( n \) distinct endpoints of arcs in \( F \), in clockwise order starting from some arbitrary point on the circle. An arc that runs clockwise from \( p_i \) to \( p_{i+1} \) for some \( i \), or from \( p_{n-1} \) to \( p_0 \), is an arc of unit length. The chromatic number of \( F \), denoted by \( \gamma(F) \), is the smallest number of colours required to colour \( F \).