A statistical average algorithm for the dynamic compound inverse problem*

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Abstract A new method named as Statistical Average Algorithm is developed for solving the structural dynamic compound inverse problem, which means to identify structural parameters with unknown input or to inverse input time history with unknown structural parameters. By taking the mechanical characteristic of the ground motion as an additional condition, an iterative algorithm based on the least-squares technique is developed so that the input process and structural parameters can be correctly determined using only output measurements. The procedure of the proposed method is discussed in detail, and a mathematical proof is presented as well in the appendix. A practical input situation such as earthquake, ambient vibration is considered in the numerical examples to verify the accuracy, reliability and robustness of the proposed algorithm. Considering the background of the practical application, both of the noise-free and noise-included output responses are considered in the numerical examples. In all cases, the proposed method identifies the structural parameters and reconstructs the input process rationally.

Keywords Structural identification, Inverse problem, Statistical average algorithm, Earthquake excitation, Ambient vibration, Buildings, Bridges

1 Introduction

The application of structural damage assessment technique in the field of civil engineering has received increasing attention in the recent years. Of particular interest is the deterioration assessment of bridges and tall buildings [1–5]. Among many nondestructive evaluation methods, the system identification techniques are very effective and appealing. However, although most of system identification methods perform well in simulation analysis, they still have several limitations that reduce their practical application. For example, most commonly used identification methods need to know both the input excitation and the output response. However, in most practical applications, the input excitation information such as ambient ground motion, wind load, etc, may not be measured precisely. Therefore, in order to identify structural parameters using a time domain algorithm, the dynamic compound problem, that is, to identify the structural parameters with unknown input or to reconstruct the input time process with unknown structural parameters, must be investigated.

In contrast to a number of publications on parameter identification with known input information, there is a paucity of publications addressed to the dynamic compound inverse problems. Initial efforts to solve this problem may be the application of the Ibrahim method, in which the input excitation was assumed as a stationary stochastic process and the output is free-decay type response. Considering that the free-decay response is hardly to be measured for actual structures, the random decrement technique was further introduced to the method [6]. The input process, however, can not be directly obtained by Ibrahim method. The Kalman filter, which incorporates both model and measurement uncertainties to achieve an optimal estimate of state variables with minimum error covariance matrix, is another possible choice to address the problem of simultaneous identification of system and excitation characteristics in some particular cases. For example, Hoshiya and Sutoh [7] developed a procedure to evaluate the parameters of a simply supported beam and characteristics of the running load on it based on the extended Kalman filter with a weighted global iteration (KF-WGI). In 1994, Benedetti and Gentile [8] presented a two-phase frequency domain method for structures subjected to ground motion, where the responses measurements at two locations of the system are assumed to be available and requirements for input information is elaborately avoided by using the ratio of the amplitude of Fourier transforms of the two recorded signals. The input information, however, can not be determined in the model parameter identification procedure. In the same year, Wang and Haldar [9] suggested a very simple method to identify structural parameters when the input excitation is unknown. In 1997, they extended their method to the situation when response measurements recorded at only a few positions are available by integrating the KF-WGI technique [10]. In the case studies of
these papers, it may be noticed that the noise level considered in the simulation examples is quite low.

Assuming the input excitation and structural parameters are both unavailable, the dynamic compound inverse problem is a new type of indirect problem of structural dynamics, which differs from the traditional load inverse problem or parameter identification problem. Therefore, a new procedure may be needed to investiate the problem. In this paper, a new method named as Statistical Average Algorithm (SAA) is proposed. In the proposed method, a statistical average operation is carried out in a transformed space and therefore constructs a key link of the algorithm. With the help of this operation, a new algorithm based on the least-squares technique is developed for the structural dynamic compound inverse problem, through which the time history of input excitation and structural parameters can be correctly determined using only the structural response measurements.

## 2 Procedure of proposed algorithm

Consider a discrete multi-degree-of-freedom linear system whose motion is governed by the set of differential equations

$$ M X(t) + C X(t) + K X(t) = F(t) $$

(1)

where $M$, $C$, $K$ is the mass, damping and stiffness matrices, each of order $n$ by $n$, $n =$ the number of DOF, $X(t) =$ the system displacement vector of order $n$, $X(t) =$ the acceleration vector of order $n$, and $F(t) =$ the force vector. It should be noticed that for a linear time-invariant system $M$, $C$ and $K$ are constants and the surveyed information at the sampling time should satisfy Eq. (1) at each instant time if there is no noise.

According to the finite element method, the global stiffness $K$ in Eq. (1) can be expressed as the superposition of the element stiffness matrix [11], which is

$$ K = \sum_{i=1}^{n} \theta_i K_i^e $$

(2)

where $\theta_i$ is the unknown stiffness parameters of element $i$, which may be the elastic modulus $E$ or $EI$. $K_i^e$ is the element stiffness matrix of element $i$ under global coordinate system after $\theta_i$ is extracted. For example, if the elastic modulus $E$ is taken as unknown parameter $\theta_i$, the matrix $K_i^e$ of a beam element will be

$$ K_i^e = \begin{bmatrix} \frac{A_i}{h_i} & 0 & 0 & -\frac{A_i}{h_i} & 0 & 0 \\ 0 & \frac{12EI}{h_i^3} & \frac{6EI}{h_i} & 0 & -\frac{12EI}{h_i^3} & \frac{6EI}{h_i} \\ 0 & \frac{6EI}{h_i^2} & \frac{4EI}{h_i} & 0 & -\frac{6EI}{h_i^2} & \frac{4EI}{h_i} \\ -\frac{A_i}{h_i} & 0 & 0 & \frac{A_i}{h_i} & 0 & 0 \\ 0 & -\frac{12EI}{h_i^3} & -\frac{6EI}{h_i} & 0 & \frac{12EI}{h_i^3} & -\frac{6EI}{h_i} \\ 0 & \frac{6EI}{h_i^2} & \frac{2EI}{h_i} & 0 & \frac{6EI}{h_i^2} & \frac{4EI}{h_i} \end{bmatrix} $$

(3)

Therefore, by introducing the finite element method, we will be lead to:

$$ K X = \sum_{i=1}^{n} \theta_i K_i^e X = \sum_{i=1}^{n} \theta_i R_{ki} $$

(4)

where $R_{ki} = K_i^e X$.

Let

$$ H_k = [R_{k1}, R_{k2}, \ldots, R_{kn}] $$

$$ \theta_k = [\theta_{k1}, \theta_{k2}, \ldots, \theta_{kn}]^T $$

(5a)

(5b)

The Eq. (4) can be further expressed as

$$ K X = H_k \theta_k $$

(6)

Similarly, for the mass and damping matrix in Eq. (1), there exists [11]

$$ M \dot{X} = H_m \theta_m $$

$$ C \ddot{X} = H_c \theta_c $$

(7)

(8)

Then by introducing the following matrix

$$ H = [H_m, H_c, H_k] $$

$$ \theta = [\theta_m, \theta_c, \theta_k]^T $$

(9)

(10)

The Eq. (1) can be rewritten as

$$ F = H \theta $$

(11)

If the response quantities $X, \dot{X}, \ddot{X}$ and input excitation $F(t)$ are available, the parameter vector $\theta$ can be directly estimated from (11) by the least-squares approximation method, that is

$$ \theta = [H^T H]^{-1} H^T F $$

(12)

However, the information of the input excitation $F$ is unknown for most practical engineering problems. Therefore, the compound inverse algorithm is introduced to solve the problem. In our work, a statistical average technique about the computational input was introduced and an iteration algorithm is developed. In the algorithm, the initial estimation of input excitation $F$ is calculated by estimated structural parameters $\theta$. Considering the estimation error in $\theta$, the $F$ must be not coincide with the real input. Therefore, an average operation is applied to the estimated input $F_0$. The proof given in the appendix of the paper will affirm that the operation will reduce the estimation error of the input. Then the average input can be taken as a new estimation of input to obtain the new estimation of structural parameters. This means that an iterative algorithm is constructed. The algorithm procedure can be further described in detail as following, where symbol ($\sim$) denotes estimated value and symbol ($) denotes the modified value.

1. Assign initial values for the unknown structural parameters. For instance, let $\theta_0 = (1, 1, \ldots, 1)$. It will be demonstrated later that the proposed algorithm is not sensitive to this initial assumption.

2. Supposing the response quantities are available at all degree of freedoms, the first estimation of the input vector $F$ can be obtained from Eq. (11).

$$ \tilde{F}_0 = H \theta_0 $$

(13)