An extended finite element method with higher-order elements for curved cracks

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Abstract A finite element method for linear elastic fracture mechanics using enriched quadratic interpolations is presented. The quadratic finite elements are enriched with the asymptotic near tip displacement solutions and the Heaviside function so that the finite element approximation is capable of resolving the singular stress field at the crack tip as well as the jump in the displacement field across the crack face without any significant mesh refinement. The geometry of the crack is represented by a level set function which is interpolated on the same quadratic finite element discretization. Due to the higher-order approximation for the crack description we are able to represent a crack with curvature. The method is verified on several examples and comparisons are made to similar formulations using linear interpolants.

Keywords Fracture, Finite elements, Crack propagation, Extended finite element method

1 Introduction

We describe the development of higher order elements within the setting of the eXtended Finite Element Method (X-FEM). The X-FEM is a numerical method to model arbitrary discontinuities in continuous bodies that does not require the mesh to conform to the discontinuities nor significant mesh refinement near singularities [2, 6, 19, 26]. In X-FEM the standard finite element approximation is enriched and the approximation space is extended by an additional family of functions. By choosing an appropriate enrichment, the extended finite element approximation space can more closely approximate the solution space for the problem considered. This type of enrichment is an application of the concept of the partition of unity [17].

For linear elastic fracture mechanics, the near tip singular stress field and the displacement discontinuity across the crack face are problematic for standard piecewise polynomial approximations. However, by adding a near tip asymptotic field and a step to the polynomial approximations we can enrich the standard FEM approximation so that good accuracy is achieved without conforming to the mesh.

This method was introduced in several papers by Belytschko and coworkers. In Belytschko and Black [2], Moës et al. [19] and Dolbow et al. [12] the crack topology was represented by an explicit discretization. Updating this explicit representation can be inconvenient when crack growth is considered. In Sukumar et al. [27] a level set representation of the topology was adopted for material interfaces. Non-planar quasi-static crack growth in three dimensions was considered in Moës et al. [20] and Gravouil et al. [16] with an orthogonal pair of level set functions to represent the crack. A PDE based method was employed to update the level sets similar to the method described in Peng et al. [22]. In all of the aforementioned papers only linear finite element approximations were used in the X-FEM approximations and in the level set interpolations. Wells et al. [34] have used the X-FEM concept in 6-node triangles in visco-plastic materials but only considered cracks that ended at an element edge.

Here we consider a technique for enriching high-order elements, and in particular quadratic finite elements. It is well known that higher-order elements provide improved accuracy for sufficiently smooth problems. Away from the crack tip this smoothness condition is satisfied in elastic problems, so improved accuracy is expected.
quadratic elements are the elements of choice for most static and quasi-static elastic problems. This is due to their higher rate of convergence, their decreased susceptibility to locking, and their ability to model curved boundaries. Furthermore, a level set interpolated by quadratic shape functions is capable of describing curved cracks; level set descriptions by piecewise linear finite elements are limited to piecewise linear cracks.

An outline of this paper is as follows. In Sect. 2, the formulation of the method is presented; the governing equations and the weak forms are given. Next, in Sect. 3 the level set representation of the crack is described. In Sect. 4, the X-FEM approximation is presented and its implementation is described. The accuracy and convergence of the method is demonstrated through several example problems in Sect. 5. Conclusions are presented in Sect. 7.

2 Formulation

2.1 Governing equations

Although many of the techniques presented here are applicable to nonlinear, large deformation problems, we present them in the context of linear elasticity. Let \( \Omega \) be a regular region bounded by a smooth curve \( \Gamma \). The boundary of the body \( \Gamma \) is the union of \( \Gamma_t \) and \( \Gamma_u \). Essential boundary conditions are imposed on \( \Gamma_u \) while traction boundary conditions are imposed on \( \Gamma_t \). Let \( u \) be the displacement and \( \varepsilon \) the strain, given by:

\[
\varepsilon = \nabla u
\]  

(1)

where \( \nabla \) indicates the symmetric part of the gradient.

If we assume that the faces of the crack \( \Gamma_{cr} \) are traction free, the strong form of the initial boundary value problem has the following form

\[
\nabla \cdot \sigma + b = 0 \quad \text{in} \quad \Omega
\]

(2)

\[
u = \bar{u} \quad \text{on} \quad \Gamma_u
\]

(3)

\[
\sigma \cdot n = \bar{t} \quad \text{on} \quad \Gamma_u
\]

(4)

\[
\sigma \cdot n = 0 \quad \text{on} \quad \Gamma_{cr}
\]

(5)

where \( \sigma \) is the Cauchy stress tensor, \( b \) the body force per unit volume, \( n \) the outward unit normal to \( \Gamma \), \( \bar{u} \) the prescribed displacement and \( \bar{t} \) the prescribed traction. The mechanical behavior of the bodies is governed by a linear elastic constitutive law:

\[
\sigma = C : \varepsilon
\]

(6)

where \( C \) is the elasticity tensor.

2.2 Weak form

We require that the trial functions \( u \) satisfy all displacement boundary conditions and have the usual smoothness properties so that \( u \) is continuous \( (C^0) \) in \( \Omega \\

\[
u \in \mathcal{U}, \quad \mathcal{U} = \{ u|u \in C^0 \text{ except on } \Gamma_{cr}, u = \bar{u} \text{ on } \Gamma_u \}
\]

(7)

The test functions \( \delta v \) are defined by:

\[
\delta v \in \mathcal{U}_0, \quad \mathcal{U}_0 = \{ \delta v|\delta v \in C^0 \text{ except on } \Gamma_{cr}, \delta v = 0 \text{ on } \Gamma_u \}
\]

(8)

The weak form of the equilibrium equation and traction boundary conditions is:

\[
\int_\Omega \sigma(u) : \varepsilon(\delta v) d\Omega
\]

\[
= \int_\Omega b \cdot \delta v d\Omega + \int_{\Gamma_t} \bar{t} \cdot \delta v d\Gamma \quad \forall \delta v \in \mathcal{U}_0
\]

(9)

Recalling the linear elastic constitutive law \( \sigma(\varepsilon) \) and the strain definition \( \varepsilon \), the following weak form of the problem can be obtained:

\[
\int_\Omega \varepsilon(u) : C : \varepsilon(v) d\Omega
\]

\[
= \int_\Omega b \cdot v d\Omega + \int_{\Gamma_t} \bar{t} \cdot v d\Gamma \quad \forall v \in \mathcal{U}_0
\]

(10)

In Belytschko and Black [2] it is shown that the above weak form is equivalent to the strong form (2–5).

3 Geometric description of crack

It is convenient, but not essential, to represent the crack by a signed distance function \( f(x) \), often called a level set. In this form, the crack is given by the zero isobar of the function \( f(x) \), i.e. by:

\[
f(x) = 0
\]

(11)

The signed distance function \( f(x) \) is defined by:

\[
f(x) = \text{sign}(n \cdot (x - \bar{x})) \min_{x \in \Gamma_{cr}} |x - \bar{x}|
\]

(12)

We only define \( f(x) \) in a subdomain around the crack as shown in Fig. 1.

In addition we define the functions \( g_i(x) \), which locate the crack tips (\( I \) runs over the number of crack tips). The function \( g_i(x) \) is given by:

\[
g_i(x) = \|x - \bar{x}^{tip}_i\|
\]

(13)