Abstract: The radial point interpolation method (RPIM) based on local supported radial basis function (RBF) and the Galerkin weak form has been developed and successfully applied to many engineering problems. Recently, a new meshfree method was proposed based on the universal moving Kriging interpolation. This paper studies the difference between the meshfree shape functions created based on the point interpolation and the Kriging interpolation. It is found that both the two methods yield the same shape function as long as the same radial basis function or semivariogram is adopted for interpolation. Although the two methods lead to the same shape function, the theorem in Kriging formulation may provide an alternative theoretical support for the RPIM. Some common semivariograms used in Kriging may also be incorporated in the RPIM. In addition, in order to satisfy the conformability requirements, a penalty technique is introduced in this paper to form a conforming Kriging, which can pass the standard patch test exactly.

Keywords: Point interpolation method, Kriging method, Radial basis functions, Meshfree method

1 Introduction

In recent years, the element-free, or meshfree method has been developed and achieved remarkable progress in computational mechanics and related fields due to its flexibility and the good convergence rate. Different versions of meshfree methods have been proposed and they have been successfully applied to the analyses of solids and structures as well as fluid flow and heat transfer problems (Liu 2002).

The element free Galerkin (EFG) method (Belytschko et al. 1994; Belytschko et al. 1996) is one of the most viable methods in which the moving least square (MLS) approach is used to construct the meshfree shape functions. Although the EFG method has been applied to many problems, there still exists some inconvenience or disadvantages when using EFG. Because the MLS shape functions lack the Kronecker delta function property, it is not easy to accurately impose essential boundary conditions in the EFG method. The Point interpolation method (PIM) has been proposed by Liu and his coworkers (Liu 2002). It uses the nodal values in the local support domain to construct the shape functions and the shape functions so constructed possess the Kronecker delta function property. According to the basis function employed, it can be mainly classified into two types, i.e., polynomial PIM (Liu and Gu 2001a, c) and radial PIM (or RPIM) (Wang and Liu 2002a, b). For the former, the moment matrix cannot always be inverted and special techniques are needed to overcome the possible singularity problem, which inevitably increases the computational cost (Liu and Gu 2001b). For the RPIM, the moment matrix is nonsingular for arbitrary nodal distributions. If the shape parameters in radial basis functions (RBF) are properly selected (Wang and Liu 2002b), the RPIM works well for practical problems. The RPIM has found applications in many engineering areas (Wang et al. 2001; Liu et al. 2003).

Although great efforts have already been spent in this area, one of the key issues in the development of meshfree methods is still to find an efficient, stable and preferably conformable method to construct the meshfree shape functions. Recently Gu (2002) found some good properties using the Kriging interpolation method and reported them in a conference abstract. Kriging is a form of generalized linear regression of an optimal estimator in a minimum mean square error sense, named after a South African mining engineer D. G. Krige. It was originally developed for mapping in the fields of geology and geophysics, mining, and photogrammetry. Kriging method has become a fundamental tool in the field of geostatistics (Olea 1999; Wackernagel 1995; Cressie 1993). It has also found applications in a variety of fields so far, including environmental monitoring and assessment.

Based on the structure and characteristics of spatially located data, Kriging method takes into account the interdependency of samples that are close to each other while allowing for a certain independence of the sample points. Apart from being an exact interpolator, it has many advantages, such as minimum mean square error, zero estimation variance, estimation interval not restricted to the data interval and so on. It also possesses good characteristics when employed to a meshless method. First the shape functions constructed by Kriging have the delta function property; hence the imposition of boundary
conditions can be easily implemented. Kriging interpolation also has the property of the partition of unity, and it can exactly reproduce any function that is included in its basis.

In this paper, Kriging based meshfree method is formulated in detail and is compared with the RPIM. It is found that the shape function obtained by the Kriging formulation is identical with that of the RPIM. However the RPIM obtains its shape function by interpolating the field variable directly using the node values in the local support domain, which is very straightforward and easy to implement in coding, while the Kriging is formulated based on the linear regression concept of an optimal estimator. In addition, the original RPIM is not a conformal method. A penalty technique is proposed to enforce the compatibility condition. The patch test is studied in detail. For completeness, a program of the present formulation has been developed in MATLAB. Several numerical examples are analyzed to demonstrate the validity and convergence of the present method.

2 Krigeing formulation

2.1 Krigeing interpolation

There are several forms of Kriging formulation, such as simple, ordinary, and general Kriging etc., all of which were initially developed for the estimation of a continuous, special attribute at an unsampled site. The ordinary Kriging method is widely used to estimate field value at a point of a problem domain, for which the variogram is known, without the prior knowledge about the mean. If sufficient nodes are given in the domain, the local Kriging is performed in which a separate Kriging system is solved at each sample point using a neighborhood of the data nodes. The local region for interpolation is also called support domain in meshfree methods. In case that the drift of the function is not a constant mean, the universal Kriging concept may be an alternative in which a polynomial drift model is commonly applied. In this paper, we discuss the interpolation method based on the universal Kriging.

Consider a field value \( u(x) \) defined in the problem domain \( \Omega \) with boundary \( \Gamma \). The domain is represented by a set of properly scattered nodes \( x_i \) \( (i = 1, 2, \ldots, N) \). \( N \) is the total number of the nodes in the whole domain. Given \( N \) field values \( u(x_1), \ldots, u(x_N) \) at field nodes \( x_1, \ldots, x_N \) in the support domain of a point \( x_0 \), we want to obtain an estimate value \( \hat{u} \) of \( u \) at \( x_0 \). Supposing that \( \hat{u} \) is linear in \( u(x_1), \ldots, u(x_n) \), it can be written as

\[
\hat{u}(x_0) = \sum_{i=1}^{n} \lambda_i u(x_i)
\]  

where \( \lambda_i \)'s are termed as weights. Suppose that \( \hat{u} \) is unbiased and the drift \( m \) is assumed as a polynomial model, the above equation is subjected to

\[
m(x) = \sum_{i=0}^{k} \mu_i p_i(x), \text{ with } p_0(x) = 1
\]  

where \( \mu_i \) is the coefficient for the polynomial basis \( p_i(x) \). For example, a linear basis in a two-dimensional problem is provided by

\[
P^T = [1, x, y], \quad k = 2
\]  

Here the superscript \( T \) stands for the transpose of the matrix. The condition that \( \hat{u} \) minimizes the mean-square prediction error \( E[(u(x_0) - \hat{u}(x_0))^2] \) requires a constrained linear optimization involving \( \lambda_1, \ldots, \lambda_n \) and Lagrange multipliers \( \mu_i \). The constrained linear optimization can be expressed in terms of the Lagrange function for the universal Kriging

\[
\begin{align*}
L(\lambda_i, \mu_i) &= E\left( (u(x_0) - \sum_{i=1}^{n} \lambda_i u(x_i))^2 \right) + 2 \mu_0 \left( \sum_{i=1}^{n} \lambda_i - 1 \right) \\
&= 2 \sum_{i=1}^{k} \mu_i \left( \sum_{j=1}^{n} \lambda_j p_i(x_j) - p_i(x_0) \right)
\end{align*}
\]  

The data are assumed to be part of a realization of an intrinsic random function \( u(x) \) with the semivariogram \( \gamma(h) \), which is in the form of

\[
\gamma(h) = \frac{1}{2} \text{Var}[u(x) - u(x + h)]
\]  

where \( \text{Var}[.] \) is the variance of the random function, \( h \) is a vectorial distance in the sampling space. Then Eq. (4) is expanded to

\[
\begin{align*}
L(\lambda_i, \mu_i) &= 2 \sum_{i=1}^{n} \lambda_i \gamma(x_i, x_0) \\
&- 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \lambda_i \lambda_j \gamma(x_i, x_j) + 2 \mu_0 \left( \sum_{i=1}^{n} \lambda_i - 1 \right) \\
&+ 2 \sum_{i=1}^{k} \mu_i \left( \sum_{j=1}^{n} \lambda_j p_i(x_j) - p_i(x_0) \right)
\end{align*}
\]  

Let \( \lambda_i \)'s be the optimal weights. The minimum mean square error is given by those weights that make all first derivatives of the Lagrangian function diminish with respect to the unknowns

\[
\begin{align*}
\frac{\partial L}{\partial \lambda_i} &= 2\gamma(x_i, x_0) - 2 \sum_{j=1}^{n} \lambda_j \gamma(x_i, x_j) + 2 \mu_0 \\
&+ 2 \sum_{i=1}^{k} \mu_i p_k(x_i) = 0 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]  

The above equation is rewritten in matrix form

\[
GW = g
\]  

where