A symmetric boundary integral formulation for cohesive interface problems

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Abstract An incremental symmetric boundary integral formulation for the problem of many domains connected by non-linear cohesive interfaces is here presented. The problem of domains with traction-free cracks and/or rigid connections are particular instances of the proposed cohesive formulation. The numerical approximation of the considered problem is achieved by the symmetric Galerkin boundary element method.

Keywords Cohesive interfaces, Boundary integral equations, Boundary elements

1 Introduction

The present work deals with isotropic linear elastic bodies linked to each other by interfaces. In particular, M non linear cohesive interfaces are considered to connect N domains made of different materials. Under the assumption of small displacements and strains, the response of such a system to quasi-static external actions is studied. The problem of N domains with M traction-free or pressurized cracks and/or rigid connections are particular instances of the proposed cohesive formulation.

The subject of the present work is significant to predict the mechanical behavior and to assess the safety factor of structures. The interface between different materials is indeed one of the most important regions governing the strength and stability of structures (Chandra Kishen, 1996) and plays a major role in fracturing of quasi brittle materials (Salvadori, 1999), polymer (Lauke and Schueller, 2001), ceramics and composites (Smith and Teng, 2001), bioengineering materials, biological solids and tissues (Middleton et al., 1996). Investigations over interface constitutive laws have been undergoing a great development in the last years. Starting from pioneering works, cohesive interfaces are often modelled by a (holonomic) nonlinear elastic relation between cohesive tractions p and opening displacement w (Hillerborg et al., 1976). This approach is meaningful only when local unloading can be reasonably assumed as negligible. During the last decade, various authors proposed non-associated elasto-plastic cohesive models to describe the interface behavior under combined normal and shear stresses in the presence of local unloading. A literature review on the subject can be found in (Salvadori, 1999).

For the problem of N domains connected by M cohesive non linear interfaces, a boundary integral incremental formulation is given (Sect. 2), in terms of the displacement fields \( u, v \), the traction field \( t \) and the displacements discontinuity field \( w \) along the interfaces. The integral operator that governs the problem, in the presence of a holonomic interface law, is proved to be linear with respect to the rate unknown fields and admits a variational formulation, its solution being a critical point of a (quadratic) functional.

Existence and uniqueness of the solution basically depend on the adopted cohesive interface law. Interface constitutive equations typically present a softening branch and this feature can cause bifurcations in the sense of multiplicity of solutions to the rate problem. This important issue has been analyzed by various authors (among others: (Maier and Frangi, 1998), (Carini and Salvadori, 2001)). In the context of the present work, uniqueness and stability issues are marginally dealt with and reference is made to interface models that are stable in the second-order work sense.

The numerical approximation of the considered problem can be achieved by different techniques. Despite the finite element method is the most widely used, boundary element methods (Bonnet et al., 1998) are very attractive for this class of problems because all non linearities are localized on the boundary of linear elastic domains. The Galerkin approximation scheme, applied to the symmetric integral formulation, ensures uniqueness, stability and convergence of the numerical solution in suitable functional spaces (McLean, 2000).

2 Single-zone formulation

Consider a homogeneous isotropic solid in a Cartesian reference system, with domain \( \Omega \subset \mathbb{R}^d, d = 2, 3 \) and with boundary \( \Gamma = \Gamma_u \cup \Gamma_p \). Assuming small strains and displacements, consider its response to quasi-static external actions: tractions \( t(x) \) on \( \Gamma_p \), displacements \( u(x) \) on \( \Gamma_u \) and domain forces \( f(x) \) in \( \Omega \).

The symmetric Galerkin boundary integral formulation of the single-zone linear elastic\(^1\) problem rests on Green’s functions with a weakly singular \( G_{uu} \), strongly singular \( G_{up} \) and \( G_{pp} \) behaviour. If the field point \( x \) is moved to the boundary in a limit process,\(^2\) Equations (1) and (2) can be easily arranged to describe a larger class of engineering problems.
“integrals” involving strongly singular kernels may be understood in their distributional nature of the Cauchy Principal Value (CPV). Similarly, the integral involving the hyper singular kernel \( G_{pp} \) can be understood in its distributional nature of Hadamard’s finite part. The boundary integral formulation of the problem formulated above reads as follows (Hong and Chen, 1998) on smooth boundaries:

\[
\frac{\partial C}{\partial t} + \int_{\Gamma_r} G_{sp}(r; l(y)) u(y) dy + \int_{\Gamma_u} G_{sp}(r; l(y)) \bar{u}(y) dy
\]

\[
= \int_{\Gamma_r} G_{su}(r; t(y)) dy + \int_{\Gamma_u} G_{su}(r; t(y)) dy + \int_{\Omega} G_{su}(r; \bar{t}(y)) dy \quad x \in \Gamma
\]

(1)

\[
D(x)t(x) + \int_{\Gamma_p} G_{pp}(r; n(x); l(y)) u(y) dy + \int_{\Gamma_u} G_{pp}(r; n(x); l(y)) \bar{u}(y) dy
\]

\[
= \int_{\Gamma_p} G_{pu}(r; n(x); t(y)) dy + \int_{\Gamma_u} G_{pu}(r; n(x); t(y)) dy + \int_{\Omega} G_{pu}(r; n(x); \bar{t}(y)) dy \quad x \in \Gamma
\]

(2)

where \( r = x - y \).

Problem (1) and (2) admits a variational formulation, i.e. it can be obtained from the stationarity of a given functional \( \Psi(u, t) \); the solution of the problem is shown to be a saddle-point (Polizzotto, 1998) for \( \Psi \). Many papers have been devoted to the numerical approximation of Eqs. (1) and (2): for a review on all related issues see Bonnet et al. (1998), McLean (2000).

3 Interfaces

Consider \( N \) domains \( \Omega^n, n \in I_N \equiv \{1, \ldots, N\} \) connected to each other by \( M \) interfaces (Fig. 1). Let \( \Gamma^{m,n} \subset \partial \Omega^n \) indicate the generic boundary pertaining to the interface \( m \in I_M \equiv \{1, 2, \ldots, M\} \). If a domain \( \Omega^n \) has nothing to do with the \( m \)-th interface, \( \Gamma^{m,n} = \emptyset \). Denoting with \( \Omega^{n_1}, \Omega^{n_2} \in I_N \) the two domains connected by the \( m \)-th interface, the two smooth boundaries \( \Gamma^{m,n_1}, \Gamma^{m,n_2} \) define the lips of the interface \( m \) (see Fig. 2). On \( \Gamma^{m,n_1}, \Gamma^{m,n_2} \) outward normals, say \( n^{n_1}, n^{n_2} \), are defined as usual.

The hypothesis of small displacements and strains implies:

\[
n^{n_1} \equiv n(x^{n_1}) = -n(x^{n_2}) \equiv -n^{n_2}
\]

and the equilibrium conditions read:

\[
t^{n_1} = \sigma(x^{n_1})n^{n_1} = -\sigma(x^{n_2})n^{n_2} \equiv -t^{n_2}
\]

It is our goal to define a reference surface, say \( \Gamma_w \), that will be identified with the interface between domains \( \Omega^{n^1} \) and \( \Omega^{n^2} \). To this aim, adopting a non linear continuum mechanics terminology, fixed \( t \) in a time interval \( T \), two one-to-one applications \( u^w, u^w \) between the reference surface \( \Gamma_w \) and the boundaries \( \Gamma^{m,n_1}, \Gamma^{m,n_2} \) are set such that,

\[
\forall x(t) \in \Gamma^{m,n_1} \exists! x \in \Gamma^m : u^w(t) = x(t) - x
\]

\[
\forall x(t) \in \Gamma^m \exists! x \in \Gamma^{m,n_1} : u^w(t) = x^{n_1}(t) - x
\]

with \( \Gamma^m \) defined by the property \( u^w(0) = u^{n_2}(0) = 0 \). In a less sophisticated way, \( \Gamma^m = \Gamma^{m,n_1} = \Gamma^{m,n_2} \) from the