A Duhamel integral based approach to one-dimensional wave propagation analysis in layered media

Abstract In this paper, a new method for analysing one-dimensional wave propagation in a layered medium is presented. It is based on Duhamel integrals in combination with the convolution quadrature method (CQM) [9, 10]. The CQM is a technique which approximates convolution integrals, in this case the Duhamel integrals, by a quadrature rule whose weights are determined by Laplace transformed fundamental solutions and a multi-step method. Duhamel integrals are used to ensure equilibrium between the layers. The methodology is closely related to structural engineering and should be more familiar to engineers in practice than the usual boundary element method. In order to investigate the accuracy and the stability of the proposed algorithm, two benchmark problems are studied. The method is presented for one-dimensional problems, namely rods, but it can be readily extended to two- or three-dimensional dynamic interaction problems, e.g., dynamic soil-structure interaction. The results are very stable with respect to time step size and they are in very good agreement with analytical solutions.

Keywords Boundary integral equations · Duhamel integral · Convolution quadrature method · One-dimensional wave propagation · Layered medium

1 Introduction

Extensive studies conducted in the past decades have demonstrated that – in general – soil-structure interaction has the following effects: (1) reduction of the resonant frequencies of systems in comparison to those of the fixed-base structure; (2) partial dissipation of the vibrational energy of the structure through wave radiation into the soil; and (3) modification of the actual foundation motion from the free field motion.

The dynamic system whose response is to be determined consists of two distinct parts with different properties: the generalised structure with bounded dimensions that consists of the actual structure together with its foundation and possibly an irregular adjacent soil region, and the unbounded soil extending to infinity. While the structure is modelled mostly either by simple multi-degree-of-freedom equivalent oscillators or by finite elements (FEM), the infinite soil modelling must contain a representation of its material and topography, and in particular, of the radiation condition [15].

In engineering practice, usually frame and plate finite elements are used to model the structure and spring and dashpot elements to model the soil. With increasing frequency, design offices are using finite elements and – in a limited number of cases – boundary elements (BEM) to model the complete soil-structure system where the soil is modelled as either a two- or a three-dimensional medium. We believe that the optimal numerical procedure will result from a hybrid FEM-BEM methodology, for harmonic excitations in a frequency domain formulation and for transient excitations in the time domain. But, although boundary integral equations (BIE) and their numerical treatment are well-known to mathematicians and engineers with theoretical background [1, 5, 4], they are rarely used in the engineering practice. The main reason is the lack of commercial boundary element computer codes, but, maybe, it is also due to the fact that engineers are not familiar with boundary integral equation formulations, or do not realize that common methodologies as the Duhamel integrals are nothing but indirect boundary integral forms.

Hence, in the present work, a new approach is described where the methodology is based on boundary
integral equations with Duhamel or convolution integrals. Duhamel integrals are commonly used in structural dynamics [7, 16] and are a powerful tool to compute the structural time domain response if the response to an unit impulse force is known. The key idea of this paper is to generalise the ideas of Duhamel integrals and combine them with a special BIE technique.

As a first approximation of general two- and three-dimensional soil-structure interaction problems in the time domain, often simple one-dimensional models are considered [17, 18]. Therefore, the novel approach is presented here for the numerical treatment of one-dimensional wave propagation, but can be extended in an analogous manner to two- and three-dimensional dynamic interaction problems. Due to the concepts familiar to the structural engineering community, the method might find wider acceptance than pure BIE techniques.

In this paper the basic equations of one-dimensional wave propagation are presented first and the boundary integral equations in time and Laplace domain are derived. Then, the new methodology is presented in its analytical form where, in order to extend the range of applicability, the Duhamel integrals are approximated by the convolution quadrature method (CQM), making use of the BIE in the Laplace domain. The CQM was developed by Lubich [9, 10] and is shortly described in the appendix A. Schanz and Antes [13, 14] applied this technique to practical engineering problems, thus demonstrating its efficiency and accuracy.

Finally, two benchmark problems are studied. The first example is a standard one, used to check the correctness and accuracy of the new formulation. In the second, more important example, a finite rod is coupled to an semi-infinite one with different material parameters. When exciting the finite rod by an impulsive force, the wave propagation in the coupled system is a one-dimensional sample for the propagation of longitudinal waves in a finite structure, which is coupled to a supporting medium extending to infinity.

2 One dimensional wave propagation problems

2.1 Time domain

The one-dimensional wave propagation in elastic media with vanishing body force, e.g., in a weightless rod, is governed by the well-known partial differential equation

\[ c^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0 \quad \text{or} \quad c^2 u'' - \ddot{u} = 0 \]  

where \( u(x, t) \) denotes the displacement which is a function of space and time. Overdots and primes indicate derivatives with respect to time \( t \) and the spatial coordinate \( x \), respectively. The scalar wave velocity \( c \) is given by

\[ c = \sqrt{\frac{E}{\rho}} \]

with \( E \) and \( \rho \) denoting Young’s modulus and the density of the material, respectively. Throughout this work, the one-dimensional media will be taken to be a weightless rod (Fig. 1).

2.2 Laplace domain

Applying a Laplace transform to equation (1) and assuming vanishing initial conditions \( u(x, t = 0) = 0 \) and \( \ddot{u}(x, t = 0) = 0 \) one obtains

\[ \mathcal{L}(u'') - \mathcal{L} \left( \frac{1}{c^2} \ddot{u} \right) = \ddot{u}'' - \frac{s^2}{c^2} \ddot{u} = 0 \]  

the Helmholtz equation in the Laplace domain. \( \ddot{u}(x, s) \) is the Laplace transform of \( u(x, t) \) and \( s \in \mathbb{C} \) the Laplace parameter.

3 Fundamental solutions and boundary integral equations

3.1 Time domain

A time domain fundamental solution for the one-dimensional wave propagation problem is solution of the inhomogeneous partial differential equation

\[ c^2 u'' - \ddot{u} = -\delta(t - \tau)\delta(x - \xi) \]  

It can easily be verified that a solution of equation (4) is given by

\[ u^e(x, \xi, t, \tau) = \frac{1}{2c} H(\tau t - c \tau - r) \]

where \( r = |x - \xi| \), and \( H \) is the Heaviside step function which guarantees the causality of the wave. The fundamental solution describes the time- and position-dependent displacements due to a Dirac step impulse applied at point \( \xi \) at time \( \tau \) in a rod of infinite length.

Now, a weighted residual statement is used. Weighting equation (1) with the fundamental solution and integrating the residual over the domain interval \([0, l]\), i.e., over the length of the rod, and over the the considered time period \([0, l]\) yields

\[ \int_0^l \int_0^l \left[ c^2 u''(x, \tau) - \ddot{u}(x, \tau) \right] u^e(x, \xi, t, \tau) dx d\tau = 0 \]  

From this equation, time dependent integral equations, relating the displacements at both ends of the rod to the