A co-rotational formulation for 3D beam element using vectorial rotational variables

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Abstract Based on a co-rotational framework, a 3-noded iso-parametric element formulation of 3D beam was presented, which was used for accurate modeling of frame structures with large displacements and large rotations. Firstly, a co-rotational framework was fixed at the internal node of the element, it translates and rotates with the node rigidly; then, vectorial rotational variables were defined, they are three smaller components of the cross-sectional principal vectors at each node, sometimes they represent different components of the cross-sectional principal vectors in incremental solution procedure so as to avoid the occurrence of ill-conditioned tangent stiffness matrix; thereafter, the internal force vector and tangent stiffness matrix in local system was derived from the strain energy of the element as its first partial derivative and second partial derivative with respect to local variables, respectively, and a symmetric tangent stiffness matrix was achieved; finally, several examples were analysed to illustrate the reliability and accuracy of this procedure.

Keywords Co-rotational procedure · Vectorial rotational variable · Large rotation · Large displacement · 3D beam element

1 Introduction

Developing an efficient beam element formulation for large displacement analysis of frame structures has been an issue of many researchers (Crisfield 1996, Hsiao et al. 1987). There already exist various formulations to meet this requirement, Hsiao et al. (1987) had divided them into three categories: Total Lagrangian formulation (Bathe and Bolourchi 1979, Kwak et al. 2001, Pai et al. 2000, Schulz and Filippou 2001), Updated Lagrangian formulation (Bathe and Bolourchi 1979, Cardona and Geradin 1988, Chen and Blandford 1991, Misra et al. 2000, Teh and Clarke 1999) and co-rotational formulation (Battini and Pacoste 2002, Crisfield 1990, Crisfield and Moita 1996, Hsiao et al. 1987, Hsiao and Lin 2000a, Teh and Clarke 1998), certainly, there also exist some mixed type formulations of them (Jiang and Chernuka 1994, Hsiao and Lin 2000b, Lin and Hsiao 2001). In addition, Simo and Vu-Quoc developed a class of geometrically-exact beam formulation, this formulation demonstrates its computational efficiency in large displacement analyses of frame structures and benefit in solving dynamic problems of flexible beam or beams system subject to large overall motions (Simo and Vu-Quoc 1986a,b, 1988, 1991, Vu-Quoc and Deng 1995, Vu-Quoc and Ebcioğlu 1995, 1996, Vu-Quoc and Simo 1987). For convenience, these formulations can also be classified into two groups: formulations with asymmetric element tangent stiffness matrices and formulations with symmetric element tangent stiffness matrices. Due to the non-commutativity of spatial rotations, most co-rotational formulations belong to the first group, and the geometrically-exact beam formulation proposed by Simo and Vu-Quoc also falls into this category (Simo and Vu-Quoc 1986a,b, 1988, 1991, Vu-Quoc and Deng 1995,
Vu-Quoc and Ebcioğlu 1995, 1996, Vu-Quoc and Simo 1987). For an asymmetric tangent stiffness matrix, more storage is occupied so as to store all its components. Simo and Vu-Quoc (1986c) denoted that in a conservative system, although their developing tangent stiffness matrix is always asymmetric, it will become symmetric once the incremental loading process arrives at an equilibrium level, Crisfield and his co-worker (Crisfield 1990, 1996, Crisfield and Moita 1996) also had found this phenomenon, so they symmetrized element tangent stiffness matrix by excluding the non-symmetric term (Simo and Vu-Quoc 1986c, Crisfield 1990, 1996, Crisfield and Moita 1996). This treatment can improve the computational efficiency greatly. Simo (1992) presented a rigorous justification for the symmetrization of non-symmetric tangent stiffness matrix. Simo (1992) and Crisfield (1996) also predicted that a symmetric tangent stiffness matrix in the co-rotational framework could be achieved if a certain set of additive rotational variables were adopted.

Up to now, numerous theoretical models of beams have been developed and applied to various practical circumstances. No single theory has proven to be general and comprehensive enough for the entire range of applications. Some beam formulations address better performance over certain class of physical problems with greater accuracy and efficiency rather than their generality, while, other models tend to a wider range of practical engineering problems, and the accuracy of the formulations has been somewhat sacrificed. In this paper, the author defined a set of vectorial rotational variables and developed an advanced co-rotational formulation for 3D beam element. In contrast with other existing beam element formulations for large displacement and large rotation analysis of frame structures, this formulation has several advantages: (1) all the variables are additive in an incremental solution procedure, this renders great simplification in updating vectorial rotational variables in incremental loading; (2) a quite simple relationship is established between the local variables and the global variables, and the transformation matrix can be derived from this relationship conveniently; (3) symmetric tangent stiffness matrices are achieved both in local system and global system; (4) total variables are used in calculating tangent stiffness matrix in local system and global system, this ensures the accuracy and reliability of beam element formulation. Considering the merits of the proposed co-rotational procedure and vectorial rotational variables, the author and his co-worker (Izzuddin and Li 2004, Li and Izzuddin 2005) have also extended them to 2D beam element, curved shell element, multi-layered tube-like beam element, laminated curved shell element and several super-elements consisting of multiple tube-like beam and shell elements in bionic structural modelling of dragonfly’s wing.

2 Description of the co-rotational framework

In this beam element formulation, several basic assumptions were adopted: (1) all elements are straight at the initial configuration; (2) the shape of the cross-section does not distort with element deforming; (3) restrained warping effects are ignored. Certainly, this co-rotational procedure and vectorial rotational variables can also be easily extended to solve some complicated beam or shell problems, such as open cross-section beams, curved beam, multi-layered composite beam and laminated curved shell element etc.

Local coordinate system and global coordinate system are illustrated in Fig. 1, both of them are Cartesian coordinate systems, where the local coordinate system is fixed at the internal node of element, and translates and rotates with the element rigid-body translation and rotation, but does not deform with the element.

In order to define the initial orientation of the local coordinate axes, an auxiliary node is prescribed, which is located in one of the symmetry plane of the element (see Point A in Fig. 1). Vectors \( v_{120} \) and \( v_{340} \) are calculated from,

\[
 v_{120} = X_{20} - X_{10} \quad v_{340} = X_{40} - X_{30} \n\]

where, \( X_{i0} (i = 1, 2, 3, A) \) is the global coordinates of Node \( i \), then the orientation vectors of local axes are defined as,

\[
 e_{x0} = \frac{v_{120}}{|v_{120}|} \quad e_{z0} = \frac{v_{120} \times v_{340}}{|v_{120} \times v_{340}|} \quad e_{y0} = e_{z0} \times e_{x0} \n\]

where, \( e_{x0}, e_{y0}, e_{z0} \) (see Fig. 1) are the normalized orientation vectors of \( x \)-axis, \( y \)-axis and \( z \)-axis in global coordinate system, respectively.

The orientation vectors \( e_{ix}, e_{iy}, e_{iz} \) of Node \( i \) at the deformed configuration are calculated from the rotational variables directly in incremental solution procedure. In particular, at Node 3 (the internal node), \( e_{3x}, e_{3y}, e_{3z} \) are coincident with the orientations of local coordinate axes,

\[
 e_{3x} = e_{x} \quad e_{3y} = e_{y} \quad e_{3z} = e_{z} \n\]

and at the initial configuration,

\[
 e_{3x0} = e_{x0} \quad e_{3y0} = e_{y0} \quad e_{3z0} = e_{z0} \n\]