An efficient semi-analytic time integration method with application to non-linear rotodynamic systems

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Abstract This paper presents a simple and efficient time-integration method for non-symmetric and non-linear equations of motion occurring in the analysis of rotating machines. The algorithm is based on a semi-analytic formulation combining powerful methods of linear structural dynamics applied to non-linear dynamic problems. To that purpose, the total solution is separated into a linear and a non-linear part, and a further partitioning into quasi-static and dynamic parts is performed. Modal analysis is applied to the undamped equations of the dynamic parts. The quasi-static parts contain all degrees of freedom, while a cost-saving modal reduction may be easily performed for the dynamic parts. Duhamel's integral is utilized for the modal equations. The time-evolution of the unknown modal excitations due to the dissipative, non-conservative, gyroscopic and non-linear effects entering Duhamel's integral is approximated during each time-step. The resulting time-stepping procedure is performed in an implicit manner, and the method is examined in some detail, in view of stability and accuracy characteristics. A rotodynamic system serves as a benchmark problem in order to demonstrate the computational advantages of the present method with respect to various other time-integration algorithms.

1 Introduction
The time-integration of dynamic systems with a large number of degrees of freedom and with inherent geometrical and/or physical non-linearities usually represents a CPU-time consuming task. This is also true if gyroscopic and non-conservative (circulatory) terms are present in the linear system parts. Direct time-integration schemes are frequently used to solve the equations of motion, since those approaches utilize the matrices of the mechanical system directly, without modal transformation, and since they can be implemented with a relative ease, see Bathe (1982). Direct integration schemes however are characterized by a certain lack of accuracy for long-term computations. Thus, a high computational effort is often necessary to achieve accurate long-term results by direct time-integration schemes. On the other hand side, a modal analysis of the linear system part is frequently required as a precondition for the determination of the appropriate time-step of direct schemes. Besides, there is a practical need for computing eigenvectors and characteristic frequencies of the linear system parts.

The present paper discusses a straight-forward but highly efficient time-integration method for non-symmetric and non-linear equations of motion, where special emphasis is given to rotodynamic systems. The semi-analytic formulation of the algorithm utilizes some of the powerful methods of linear structural dynamics, like modal analysis. For that sake, the total solution is separated into a linear and a non-linear part, and a further partitioning into quasi-static and dynamic parts is performed. Due to this formal linearity, modal analysis can be introduced consistently into the non-linear problem, and Duhamel's integral can be used to account for the excitation of the modal oscillators. While all physical degrees of freedom are considered for the quasi-static solution parts, a cost-saving modal reduction can be easily performed for the dynamic parts after modal transformation. The unknown excitations due to the dissipative, non-conservative, non-symmetric and non-linear forces are approximated in their time-evolution. These approximations are performed for the evaluation of Duhamel's integral only, and do not represent an approximation of the total solution. Thus the error of the approximation is smoothened by time-integration. In the present paper, the time-evolution of unknown terms in Duhamel's integral is approximated by suitable linear or quadratic functions, forming a family of approximate methods. As Duhamel's integral is used in this procedure, the transfer matrix for the present method is based on the exact transfer matrix of the linear modal oscillators. The computations are considerably simplified by the above partitioning into various solution parts and by the introduction of modal analysis. The resulting time-stepping procedure is implicit, and will be discussed in some detail under the view of stability and accuracy. Since time steps to be considered in a dynamic analysis are small, a direct iteration scheme usually suffices for the implicit procedure. For some special kinds of non-linearities, e.g. for bilinear, quadratic or cubic restoring forces, and in the case of single degree of freedom systems, the computation of the non-linear dynamic response could be performed even analytically, without any iteration.

A basic feature of the present semi-analytic algorithm is the separation of the solution of the differential equations into the quasi-static and dynamic parts. For continuous structural systems, this separation of the solution was introduced by Mindlin and Goodman (1950) and was used
by Boley and Barber (1957) for problems of thermoelasticity. These authors applied the idea to the thermal bending of plates, where it has been found that the accuracy and convergence of this method could be considerably improved due to the separation of the solution. Based on these results, Irschik (1986) and Irschik and Ziegler (1988) extended the approach to physically non-linear vibrations of continuous structures, introducing an additional separation of the solution into a linear and a non-linear part, see Irschik and Ziegler (1995) for a review, and Ziegler, Irschik and Holl (1995) for non-linear wave propagation.

The present paper gives the extension of the above cited contributions to discretized dynamic systems in the presence of non-symmetric matrices and circulatory forces. It is furthermore demonstrated, that the separation of the solution into the above mentioned parts results in an increase in the accuracy of the result.

A further crucial feature of the present contribution is the consistent use of modal analysis. In literature modal analysis for linear systems with non-symmetric matrices is usually performed considering the corresponding system of differential equations of first order, so that a preceding matrix manipulation becomes necessary. The computation of the eigenvalue problem then involves a higher effort, since computations with complex numbers become necessary for the system of the dimension $2n$, where $n$ denotes the number of degrees of freedom. Based on a reformulation of the second order differential equations of motion, the present algorithm avoids the computation with matrices of the dimension $2n$. A modal analysis is thus performed, which takes into account the linear symmetric parts and a Rayleigh-type damping of the equations of motion. The method involves real eigenvalues as well as computations with matrices of the dimension $n$ only. In the presented algorithm, furthermore, a reduction to $m < n$ modal degrees of freedom can be easily performed, neglecting the influence of higher modes of the dynamic solution part. For the representation of the dynamic response of the system, usually only the first few modes have to be considered, since the dynamic solution part usually acts like a band pass filter. In contrast, the static part of the solution has to be computed using all $n$ degrees of freedom.

Comparing the present modal reduction method to the well-known mode acceleration method, see Craig (1981), Cornwell et al. (1983), and to the mode displacement method, see Salmonte (1982), Leger and Wilson (1988), it is noted that the modal transformation is applied to the total solution in these latter formulations, and a type of quasi-static solution part is introduced afterwards on a more intuitive basis, which is motivated by the need of getting $m < n$ sufficiently small in the modal reduction procedure. Note that the mode acceleration method and the mode displacement method have not been introduced in the context of non-symmetric and non-linear systems, but concentrate to symmetric and linear systems with Rayleigh-type damping. It can be shown that the present method is superior for the latter linear systems, which will be reported elsewhere.

Modal analysis in its usual form of course is not directly applicable to non-linear systems but to their linear system part only, or for the case of small vibrations superimposed upon a given state. It seems to be obvious that the present algorithm works well for a system, where the non-symmetric and non-linear terms represent a small perturbation of the linear symmetric system. It is demonstrated below, however, that the algorithm is superior to conventional direct time-integration algorithms also in the case of large non-symmetric elements of the matrices and for a considerable amount of non-linearity.

With respect to linear systems, the present algorithm of course gives exact results for linear problems with symmetric matrices and a Rayleigh type damping matrix. Hence, an arbitrarily large time step may be used in such a case. In contrast, modifications of the approximate Newmark method have been introduced in the literature for systems of this type, see e.g. Fung (1997) for unconditionally stable methods of a higher order of accuracy. Note furthermore, that the present algorithm is similar to the transfer matrix method presented in Weaver et al. (1990), Gasch and Knothe (1987) and Schwibinger and Nordman (1988) for linear problems with symmetric matrices and a Rayleigh type damping matrix. In these contributions, however, no separation of the solution into quasi-static and dynamic parts has been used and no extensions to non-symmetric and non-linear problems have been considered there.

Since the present method is exact for linear symmetric systems, the numerical stability of the algorithm needs to be demonstrated only for non-symmetric and non-linear problems. In the literature, discussions of the stability of numerical algorithms are usually performed for linear and symmetric systems only. In Hughes (1983) the stability is demonstrated for some direct numerical algorithms and additionally a definition of the numerical dissipation and dispersion, and of the local and global error is given. Further interesting features, like overshoot, are presented in Hilber and Hughes (1978), where the HHT-$x$-method is used. For the computation of adaptive time-steps some error measures have been developed for different systems in order to further increase the computational efficiency by computing an appropriate time-step, see Chung and Belytschko (1998), Hulbert and Hughes (1990), Kulkarni, Belytschko and Bayliss (1995), Smolinski, Sleith and Belytschko (1996).

In the present paper, these measures are extended to the case of non-symmetric damped linear systems for some members of the present family of algorithms. Some portions of the linear damping forces are treated as state-dependent excitation forces of the linear system, and the presented numerical algorithm is applied to this quasi-non-linear problem. The resulting approximate numerical procedures are reformulated to a form suitable for the above cited stability considerations. All of the presented algorithms turn out to have at least an order of convergence of two, some of them being only conditionally stable in the examples under consideration. Analogously, numerical dispersion and dissipation measures are calculated, where the same order of convergence is shown to result. For special cases, such as quadratic or cubic restoring forces, this type of analysis of stability and accuracy can be extended also to non-linear systems, which will