An alternative version of the Pian–Sumihara element with a simple extension to non-linear problems

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Abstract The stiffness matrix for the Pian–Sumihara element can be obtained in a different way than originally presented in Pian and Sumihara (1984). Instead of getting the element matrix from a hybrid stress formulation with five stress terms one can use a modified Hu–Washizu formulation using nine stress and nine strain terms as well as four enhanced strain terms. Using orthogonal stress and strain functions it becomes possible to obtain the stiffness matrix via sparse B-matrices so that numerical matrix inversions can be omitted. The advantage of using the mixed variational formulation with displacements, stresses, strains, and enhanced strains is that the extension to non-linear problems is easily achieved since the final computer implementation is very similar to an implementation of a displacement element.

1 Introduction
The development of hybrid stress finite elements was initiated by Pian in (1964). Among the following publications on hybrid elements [see e.g. Pian and Tong (1969, 1986), Rubinstein et al. (1983), Puch and Atluri (1984), Xue et al. (1985), Pian and Wu (1988)] the paper of Pian and Sumihara (1984) defined a milestone in finite element history. The Pian–Sumihara element (P-S) performs very well and other authors have tried to utilize the basic ideas of the P-S element for the construction of new elements. The Pian–Sumihara element can be derived on the basis of the Hellinger–Reissner principle and the use of incompatible displacements \( u_i \):

\[
\Pi = \int_V \left[ -\frac{1}{2} \sigma^T E^{-1} \sigma - u^T f + \sigma^T (D \mathbf{u} + D \mathbf{u}_i) \right] \, dV \\
- \int_S \delta \mathbf{u}^T T \, dS
\]  

Carrying out the variation in (1) we get

\[
\int_V \delta \sigma^T \left[ -E^{-1} \sigma + D (\mathbf{u} + \mathbf{u}_i) \right] \, dV = 0 
\]

\[
\int_V \delta (D \mathbf{u})^T \sigma \, dV = \int_V \delta \mathbf{u}^T f \, dV + \int_S \delta \mathbf{u}^T T \, dS 
\]

\[
\int_V \delta (D \mathbf{u}_i)^T \sigma \, dV = 0 
\]

The use of the original Wilson incompatible shape functions \( N^i_1 = 1 - \zeta^2 \) and \( N^i_1 = 1 - \eta^2 \) does not allow an exact satisfaction of the constraint Eq. (4). Equation (4) means that the strains from the incompatible displacement field should be orthogonal to the assumed stresses.

If we choose the incompatible functions in the form

\[
N^i_1 = \left( 1 - \frac{J_2}{J_0} \eta \right) (1 - \zeta^2) + \frac{J_1}{J_0} \zeta (1 - \eta^2) 
\]

\[
N^i_2 = \left( 1 - \frac{J_1}{J_0} \zeta \right) (1 - \eta^2) + \frac{J_2}{J_0} \eta (1 - \zeta^2) 
\]

the resulting strains are orthogonal to the Pian–Sumihara stresses

\[
\sigma = \begin{bmatrix} 1 & 0 & 0 & a_2^2 \zeta^2 & a_1^2 \eta^2 \\ 0 & 1 & 0 & b_2^2 \zeta^2 & b_1^2 \eta^2 \\ 0 & 0 & 1 & a_2 b_2 \zeta & a_1 b_1 \eta \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \beta_5 \end{bmatrix} = SB
\]

where \( a_i, b_i, J_0, J_1, J_2 \) are obtained from the transformation between the Cartesian coordinates \( x, y \) and the natural element coordinates \( \zeta, \eta \):

\[
\begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{4} \sum_{i=1}^4 (1 + \xi_i \zeta)(1 + \eta_i \eta) \begin{bmatrix} x_i \\ y_i \end{bmatrix}
\]

\[
\begin{bmatrix} a_0 & b_0 \\ a_1 & b_1 \\ a_2 & b_2 \\ a_3 & b_3 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \end{bmatrix}
\]

\[
J_0 = a_1 b_2 - a_2 b_1; \quad J_1 = a_1 b_3 - b_1 a_3; \quad J_2 = a_3 b_2 - a_2 b_3
\]

Received 31 January 2000

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The author would like to thank Professor T.H.H. Pian for reading the manuscript and his time for valuable discussion.
Instead of choosing incompatible displacements $u$, it is possible to choose directly so called “enhanced strains” [see e.g. Simo and Rifai (1990), Simo and Armero (1992), Simo et al. (1993), Andelfinger et al. (1992), Andelfinger and Ramm (1993), Yeo and Lee (1996), Bischoff et al. (1999), Crisfield et al. (1995), FreiSchläger and Schweizerhof (1996), Glaser and Amero (1995), Korelc and Wriggers (1996), Wriggers and Korelc (1996), Nagtegaal and Fox (1996), Wriggers and Reese (1996), Roehl and Ramm (1996), Piltner and Taylor (1995, 1999, 2000)] which can be constructed for the four node element by using the Wilson incompatible shape functions. Instead of evaluating the exact derivatives in cartesian coordinates an approximation is used which involves the Jacobian $J(\xi, \eta)$ evaluated at the center of the element [see Taylor et al. (1976)]. Therefore instead of (1) we can use the following functional:

$$\Pi = \int_V \left[ -\frac{1}{2} \sigma^T E^{-1} \sigma - u^T f + \sigma^T (D\dot{u} + \epsilon^i) \right] dV$$

$$- \int_S u^T \mathbf{T} dS$$

(11)

where $\epsilon^i$ is the enhanced strain field derived using an approximated differential operator matrix $D^0$

$$\epsilon^i = D^0 u$$

(12)

For the four node element the enhanced strains derived from the Wilson incompatible shape functions take the form

$$\epsilon^i = B^i \dot{\lambda}$$

(13)

The constraint equation equivalent to (4) is given as

$$\int_V \delta \epsilon^i \sigma^T dV = 0$$

(14)

and is identically satisfied for the Pian–Sumihara stresses (7) and the enhanced strains (13).

Both functionals (1) and (11) lead to the same element stiffness matrix when using the assumed fields for $\sigma$, $u$, and $\epsilon^i$ explained above. The stiffness matrix of the Pian–Sumihara element has the form

$$k = L^T H^{-1} L$$

(15)

where

$$H = \int_V S^T E^{-1} S dV$$

(16)

$$L = \int_V S^T B dV$$

(17)

In the notation used, $B$ is the strain matrix resulting from the compatible displacement field $u = Nq$ and the associated strains can be written as

$$Du = Bq$$

(18)

where $D$ is a linear differential operator matrix. For an efficient implementation of the Pian–Sumihara element into a finite element program we refer to Chapter 13 in Zienkiewics and Taylor (1989).

2 Mixed finite element formulation

Because the Pian–Sumihara element shows an excellent behavior in linear problems, it is desirable to extend these features to non-linear problems. The use of the hybrid stress method for non-linear problems has been discussed in the literature by several authors [see e.g. Atluri (1973), Murakawa and Atluri (1978), Pian (1976), Boland and Pian (1977), Reed and Atluri (1983), Simo et al. (1989), Seki and Atluri (1994)]. It appears that for non-linear problems it is more convenient to include assumed strains into the formulation. In the following a mixed variational formulation with enhanced strains is considered for an alternative way to compute the stiffness matrix of the Pian–Sumihara element and to include the treatment of non-linear problems. Felippa mentions in his work the possibility of obtaining the same element stiffness matrix with different element formulations [see e.g. Felippa and Militello (1998)]. Here we consider mixed variational formulations with enhanced strains.

The use of two different versions of enhanced strain methods became quite popular during recent years. Depending on the choice of the enhanced strains, it becomes possible to achieve element performance similar to the performance of the Pian–Sumihara element. The original enhanced strain method was introduced by Simo and Rifai in 1990 and is based on the following functional

$$\Pi = \int_V \left[ \frac{1}{2} \epsilon^T \epsilon - u^T f \right] dV - \int_S u^T \mathbf{T} dS - \int_V \sigma^T \epsilon^i dV$$

(19)

where

$$\epsilon = Du + \epsilon^i$$

(20)

and $\epsilon^i$ is the assumed enhanced strain field.

An alternative version of enhanced strains is based on the following modified Hu–Washizu formulation:

$$\Pi(u, \epsilon, \epsilon^i, \sigma) = \int_V \left[ \frac{1}{2} \epsilon^T \epsilon - u^T f \right] dV - \int_S u^T \mathbf{T} dS$$

$$- \int_V \sigma^T (\epsilon - Du - \epsilon^i) dV$$

(21)

This functional has been used by Piltner and Taylor (1995, 1999) to introduce a very fast implementation of mixed elements with the aid of sparse B-matrices. Both versions of enhanced strains can be extended to non-linear problems.