Solving three-dimensional layout optimization problems using fixed scale wavelets

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Abstract The layout optimization problem in three-dimensional elasticity is solved with a meshless, wavelet-based solution scheme. A fictitious domain approach is used to embed the design domain into a simple regular domain. The material distribution and displacement field are discretized over the fictitious domain using fixed-scale, shift-invariant wavelet expansions. A discrete form of the elasticity problem is solved using a wavelet-Galerkin technique during each iteration of the layout optimization sequence. Approximate solutions are found with an efficient preconditioned conjugate gradient (PCG) solver using non-diagonal preconditioners which lead to convergence rates that are insensitive to the level of resolution. The convergence and memory management properties of the PCG algorithm make the analysis of large-scale problems possible. Several wavelet-based layout optimization examples are included.

1 Introduction

The goal of a typical layout or topology optimization problem is to determine the layout of the most rigid structure capable of supporting a given load, constrained by the amount of material available and restricted spatially to be within a prescribed package space. An important benefit of this formulation is that the general shape and connectivity of the design are not specified a priori. Because of this, layout optimization has become an integral part of conceptual design in several engineering contexts. Engineers use layout optimization to aide in the design of mechanical components creating stronger, lighter, and more cost effective designs.

Layout optimization in three dimensions often requires the use of large-scale models solved using iterative optimization algorithms. During each iteration of the optimization sequence the governing equations (e.g., elasticity) are solved to determine the performance of the current design. Typically, finite element methods are used for this purpose and one finite element is used per “shape” design variable. Thus, when a high resolution representation of the shape is desired, a very large number of elements is required. This implies that when layout optimization is used in a detailed design setting, very large-scale finite element models must be solved numerous times (at least once per optimization iteration). As the problem size increases, the memory requirements and processing time necessary to obtain finite element solutions dominate the overall process. For large scale problems, high computational demands often make the problems impractical to solve using reasonable computational resources.

Iterative solvers (e.g., based on conjugate gradients) are required to solve the discretized elasticity equations arising in layout optimization problems of typical size. Unfortunately, when finite elements are used, the condition number of the system matrices often increases with the size of the problem. When this happens, as the problem size becomes very large, iterative schemes may fail to converge due to poor conditioning (Stephane, 1992). This problem becomes particularly acute when the material distribution in the domain is very heterogeneous, as is the case in layout optimization problems. In such cases, the amount of shape detail available in the design domain is limited by the analysis technique. This paper presents an alternative method to solve layout optimization problems which replaces finite element analysis with meshless, wavelet-based techniques that are specially tailored to solve these problems efficiently.

1.1 Outline of solution strategy

The recent literature suggests a number of meshless methods that may be used address the issue at hand. For example, the smooth particle hydrodynamics method (Lucy, 1977; Gingold and Monagham, 1977); the diffuse element method (Nayroles et al., 1992); the element-free Galerkin method (Belytschko et al., 1994; Belytschko et al., 1995; Krysl and Belytschko, 1995); the Petrov-Galerkin diffuse element method (Krongauz and Belytschko, 1997); the reproducing kernel particle method (Liu et al., 1996; Liu et al., 1997); $h$-$p$ clouds (Duarte and Oden, 1996) and the meshless local Petrov-Galerkin method (Atluri and Zhu, 1998a, b; Zhu and Atluri, 1998) are a few possible choices. Our approach follows the previous work by Diaz (1999) and DeRose and Diaz (1999) and relies heavily on the work of Wells and Zhou (1992), Glowinski et al. (1995, 1996a, b), Kunothe (1995), Dumont and Lebon (1996), and Barsch et al. (1997).

In our approach, discretized models are constructed starting from an image of the component embedded into a simple fictitious domain, i.e., a square in two dimensions.
and a cube in three dimensions. In two dimensions, this process corresponds to a pixel level discretization of the component, while in three dimensions the discretization corresponds to a voxel discretization (the three-dimensional equivalent to pixels). Given the image discretization, wavelet bases defined at the resolution of the component image are introduced in a standard Galerkin scheme. An iterative, preconditioned conjugate gradient solver is used to solve the resulting linear equations. A special preconditioner that results in convergence rates that are insensitive to the resolution of the image discretization is used. This approach is easily integrated within traditional layout optimization techniques and requires minimal modifications to standard layout optimization algorithms.

2 Problem statement

In a typical layout optimization problem, the goal is to determine the optimal material distribution within a design domain given a fixed amount of material. Figure 1 illustrates the setting of a typical problem. Here, \( \omega \) represents the design domain or package space where the structure is to be laid, \( \Gamma^u \subset \partial \omega \) is the section of the boundary where displacements are constrained and \( \Gamma^t \subset \partial \omega \) is the section of the boundary where loads (tractions) are applied. As usual, \( \Gamma^u \cup \Gamma^t = \partial \omega \) and \( \Gamma^u \cap \Gamma^t = \emptyset \).

A typical formulation of the layout optimization problem may be stated as:

Find \( \rho(\mathbf{x}) \) that

\[
\text{minimizes } \int_{\Gamma^t} t u_{\omega} \, d\omega \\
\text{subject to } \int_{\omega} \rho(\mathbf{x}) \, d\omega \leq V \times \text{meas}(\omega) \\
0 < \rho(\mathbf{x}) \leq 1; \quad \mathbf{x} \in \omega \tag{1}
\]

where \( \rho(\mathbf{x}) \) describes the material distribution in the package space \( \omega \in \mathbb{R}^n \); \( V, 0 < V < 1 \), limits the amount of available material; \( t \) is the applied traction; and \( u_{\omega} \) is the equilibrium displacement field, i.e., the solution to the following elasticity problem:

Find \( u_{\omega} \in K_{\omega} \) that

\[
\text{minimizes } \Pi_{\omega}(u_{\omega}) = \frac{1}{2} \int_{\omega} E_{\omega} \varepsilon(u_{\omega}) \varepsilon(u_{\omega}) \, d\omega \\
- \int_{\Gamma^t} t u_{\omega} \, d\Gamma \tag{2}
\]

In (2),

\[
K_{\omega} = \{ u_{\omega} \in H^1(\omega): u_{\omega} = 0 \, \text{on } \Gamma^u \subset \partial \omega \}
\]

is the set of all kinematically admissible solutions. For three-dimensional problems, \( u_{\omega} = (u_{\omega_1}, u_{\omega_2}, u_{\omega_3}) \), \( t = (t_1, t_2, t_3) \), and \( E_{\omega}(x,y,z) \) is the tensor of elastic properties which depend on the current design \( \rho(\mathbf{x}) \). As in typical in layout optimization problems (e.g., Bendsoe and Kikuchi, 1988) \( \rho(\mathbf{x}) \) represents the effective density of material assigned to a point \( \mathbf{x} \) in the package space \( \omega \). Note that the notations \( (x,y,z) \) and \( (x_1,x_2,x_3) \) are used interchangeably.

2.1 Fictitious domain solution method

To solve problem (2) using a fictitious domain approach we first embed the package space \( \omega \) into a simplified domain \( \Omega = [0,d] \times [0,d] \times [0,d] \), as shown in Fig. 2 (shown in two dimensions for simplicity). The domain \( \Omega \), called the fictitious domain, becomes the domain in a new \( \Omega \)-periodic elasticity problem,

Find \( u \in K_{\Omega} \) that

\[
\text{minimizes } \Pi_{\Omega}(u) = \frac{1}{2} \int_{\Omega} E_{\Omega} \varepsilon(u) \varepsilon(u) \, d\Omega - \int_{\Omega} f u \, d\Omega \tag{4}
\]

In (4),

\[
K_{\Omega} = \{ u_i \in V_{\Omega}: B u_i = 0 \, \text{on } \Gamma^u \subset \partial \omega \} \tag{5}
\]

and

\[
V_{\Omega} = \{ u_i \in H^1(\Omega): u_i \text{ is } \Omega\text{-periodic} \} \tag{6}
\]

where \( B \) is a linear operator on \( V_{\Omega} \), \( u = (u_1, u_2, u_3) \), \( f = (f_1, f_2, f_3) \), and \( E_{\Omega}(x,y,z) \) is the material tensor over the entire fictitious domain \( \Omega \). Due to the periodic setting of problem (4), the material tensor \( E_{\Omega} \) and forcing function \( f \) must be \( \Omega \)-periodic. To recast the original elasticity problem (2) into the fictitious domain problem (4), one must:

(i) Define an appropriate material tensor \( E_{\Omega} \) over the entire domain \( \Omega \). Typically, \( E_{\Omega} < E_{\omega} \), i.e., the fictitious material is very compliant.

Fig. 1. The setting of a typical layout optimization problem

Fig. 2. The package space embedded into a fictitious domain