SMALL COMPLETE ARCS IN PROJECTIVE PLANES

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In the late 1950’s, B. Segre introduced the fundamental notion of arcs and complete arcs [48, 49]. An arc in a finite projective plane is a set of points with no three on a line and it is complete if cannot be extended without violating this property. Given a projective plane \( \mathcal{P} \), determining \( n(\mathcal{P}) \), the size of its smallest complete arc, has been a major open question in finite geometry for several decades. Assume that \( \mathcal{P} \) has order \( q \), it was shown by Lunelli and Sce [41], more than 40 years ago, that \( n(\mathcal{P}) \geq \sqrt{2q} \). Apart from this bound, practically nothing was known about \( n(\mathcal{P}) \), except for the case \( \mathcal{P} \) is the Galois plane. For this case, the best upper bound, prior to this paper, was \( O(q^{3/4}) \) obtained by Szönyi using the properties of the Galois field \( GF(q) \).

In this paper, we prove that \( n(\mathcal{P}) \leq \sqrt{q} \log^c q \) for any projective plane \( \mathcal{P} \) of order \( q \), where \( c \) is a universal constant. Together with Lunelli-Sce’s lower bound, our result determines \( n(\mathcal{P}) \) up to a polylogarithmic factor. Our proof uses a probabilistic method known as the dynamic random construction or Rödl’s nibble. The proof also gives a quick randomized algorithm which produces a small complete arc with high probability.

The key ingredient of our proof is a new concentration result, which applies for non-Lipschitz functions and is of independent interest.

1. Introduction

A projective plane of order \( q \) consists of a set of \( q^2 + q + 1 \) points and a set of \( q^2 + q + 1 \) lines, where each line contains exactly \( q + 1 \) points and two distinct

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points lie on exactly one line. It is easy to deduce from the definition that each point is contained in exactly $q + 1$ lines and two distinct lines have exactly one common point. Finite projective planes are fundamental objects in combinatorics and several related areas such as coding theory; for more details about projective planes we refer to [25] and [32]. Throughout the paper, we assume that our projective plane has order $q$.

The most important example of a projective plane is, perhaps, the Galois plane $PG(2,q)$, constructed as follows. Let $V$ be the vector space of dimension three over the Galois Field $GF(q)$, where $q$ is a prime power. The points and lines of the projective plane $PG(2,q)$ are the 1 and 2 dimensional subspaces of $V$, respectively, with the natural inclusion. For large $q$, there are many planes of order $q$ not isomorphic to $PG(2,q)$. On the other hand, it is not known whether there exists a projective plane of order not a prime power.

In the late 1950’s, B. Segre ([48,49]) introduced the notions of arcs and complete arcs. An arc in a plane is a set of points with no three on a line and maximal arcs under the set inclusion are called complete arcs. A line containing two points of an arc is called a secant. By definition, an arc is complete if and only if its secants cover the whole plane. Segre [48] asked the following fundamental question:

**In a plane $\mathcal{P}$ of order $q$, how many points can a complete arc have?**

Since Segre’s introduction, this question has become one of the main research topics in finite geometry. Especially, the problems of finding the maximum and minimum possible sizes of complete arcs and characterizing those complete arcs have attracted much attention.

For the maximum size, it is fairly trivial that an arc cannot have more than $q+2$ points (the reader may consider it an easy exercise). Segre himself proved that for odd $q$, an arc of $PG(2,q)$ has at most $q + 1$ points, and the maximum is attained if and only if the arc is a conic, which is basically the set of points $(x,y,z)$ satisfying $xz = y^2$. (Of course, with respect to the definition of $PG(2,q)$ given above, a point $(x,y,z)$ actually means the one dimensional subspace generated by $(x,y,z)$.) For even $q$, the maximum is $q+2$, but the characterization of all such arcs is still not completed [10]. The second largest cardinalities have been studied too (see e.g. [27–29,10,53] and references therein).

The other direction, the minimum possible size, seems to be more interesting from the combinatorial point of view since it is a mini-max question. Given a plane $\mathcal{P}$ of order $q$, we denote by $n(\mathcal{P})$ the size of a smallest complete arc in $\mathcal{P}$. The main goal of this paper is to give a nearly sharp estimate