Let $G$ be a $k$-connected graph $G$ having circumference $c \geq 2k$. It is shown that for $k \geq 2$, then there is a bond $B$ which intersects every cycle of length $c - k + 2$ or greater.

1. Introduction

It was shown by [7] that for a loopless 2-connected graph $G$ with circumference $c$ and cocircumference $c^*$, it holds that $|E(G)| \leq \frac{1}{2}cc^*$. Recently, Lemos and Oxley [2] showed that this bound holds not only for graphs but for connected matroids in general. They showed that if $M$ is a connected matroid with at least 2 elements, and $M$ has circumference $c$, and cocircumference $c^*$, then $|e(M)| \leq \frac{1}{2}cc^*$.

Oxley [5] posed the following conjecture:

Conjecture 1.1. For any connected matroid $M$ with at least 2 elements, one can find a collection of at most $c^*(M)$ cycles which cover each element of $M$ at least twice.

In [4], Neumann-Lara et al showed that the above conjecture holds for cographic matroids. They used the following lemma which appears in Wu [7].

Lemma 1.1. Let $G$ be a 2-connected graph. Then there is a bond which intersects every cycle of length $c$ or $c-1$.

In [3], a corresponding result for cycles was proven, namely:

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Theorem 1.1. Let $G$ be a $k$-connected graph with cocircumference $c^*$. Then for $k \geq 2$, there is a cycle which intersects every bond of size $c^* - k + 2$ or greater.

The object of this paper is to dualize this result. We first prove the following theorem which constitutes the bulk of this paper.

Theorem 1.2. For a $k$-connected graph where $k \geq 2$ and $c = c(G) \geq 2k$, if $C_1$ and $C_2$ are a pair of cycles which intersect in at most one vertex, then it holds that $|V(C_1)| + |V(C_2)| \leq 2(c - k + 1)$.

Using this result, we shall prove:

Theorem 1.3. For any $k$-connected graph $G$ where $k \geq 2$ and having circumference $c \geq 2k$, there is a bond $B$ which intersects every cycle of length $c - k + 2$ or greater.

2. Disjoint path lemmas

A useful tool for $k$-connected graphs is the so-called ‘Fan Lemma’ (see [1]). We shall use the following variant of this lemma:

Lemma 2.1. Let $G$ be a $k$-connected graph where $k \geq 1$, let $X$ and $Y$ be disjoint subsets of vertices of a graph $G$ where $|X| \geq k$, and $|Y| \geq k$. There exist $k$ vertex-disjoint paths $P_1, \ldots, P_k$ each originating at a vertex in $X$ and terminating at a vertex in $Y$, and each path intersecting $X$ and $Y$ only at its terminal vertices.

We shall need a modified version of the above lemma, namely:

Lemma 2.2. Let $G$ be a $k$-connected graph where $k \geq 1$, let $X$ and $Y$ be disjoint sets of vertices where $0 < |X| \leq k$ and $|Y| \geq k$. Assume $X = \{u_1, u_2, \ldots, u_s\}$ and let $w_1, \ldots, w_s$ be positive integers such that $\sum_{i=1}^{s} w_i = k$. Then there exist $k$ internally vertex-disjoint paths from $X$ to $Y$ such that for each $i$, exactly $w_i$ of these paths originate at $u_i$. Moreover, no two paths terminate at the same vertex in $Y$, and each path intersects $X$ and $Y$ only at its terminal vertices.

3. Finding large independent sets

In this section, we shall show that if the circumference of a $k$-connected graph is ‘small’, then it must contain a ‘large’ independent set of vertices.