NOTE

PACKING NON-RETURNING A-PATHS*

GYULA PAP†

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Chudnovsky et al. gave a min-max formula for the maximum number of node-disjoint non-zero A-paths in group-labeled graphs [1], which is a generalization of Mader’s theorem on node-disjoint A-paths [3]. Here we present a further generalization with a shorter proof. The main feature of Theorem 2.1 is that parity is “hidden” inside $\nu$, which is given by an oracle for non-bipartite matching.

1. Introduction

W. Mader [3] gave a theorem on packing A-paths, which was re-stated in a more transparent form in two papers, in [5] by A. Sebő and L. Szegő, and independently in [1] by M. Chudnovsky, J. Geelen, B. Gerards, L. Goddyn, M. Lohman and P. Seymour. In [1] a min-max formula is given for the packing of non-zero A-paths in group-labeled graphs, which contains Mader’s theorem as a special case. In this paper we show Theorem 2.1 on non-returning A-paths in “permutation-labeled” graphs, which contains the result on non-zero A-paths. The method of proof in this paper is also related to the short proof of Mader’s theorem given by A. Schrijver [4]. We use an analogue of Berge’s alternating paths’ lemma in the proof of the main theorem of this paper.

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†The author is a member of the Egerváry Research Group (EGRES).
Let \( G = (V, E) \) be an oriented graph with node-set \( V \), arc-set \( E \) and a fixed set \( A \subseteq V \) of terminals. The orientation of arcs is needed only for reference. Let \( \Omega \) be an arbitrary set of “potentials” and let \( \omega : A \to \Omega \) define the potential of origin for the terminals. Let \( \pi : E \to S(\Omega) \) where \( S(\Omega) \) is the set of all permutations of \( \Omega \). For an arc \( ab = e \in E \), let \( \pi(e, a) := \pi(e) \) and \( \pi(e, b) := \pi^{-1}(e) \) be the mapping of potential on arc \( ab \). An \( A \)-path in \( G \) is a sequence of nodes and arcs which correspond to a path in the underlying undirected graph joining two distinct nodes of \( A \), not using any other node in \( A \). For an \( A \)-path \( P = (v_0, e_0, v_1, e_1, \ldots, e_{k-1}, v_k) \), let \( \pi(P) := \pi(e_0, v_0) \circ \pi(e_1, v_1) \circ \cdots \circ \pi(e_{k-1}, v_{k-1}) \) define the mapping of potentials on \( P \). Let \( P \) be called non-returning if \( \pi(P)(\omega(v_0)) \neq \omega(v_k) \). In other words, an \( A \)-path is returning if it maps the potentials of origin onto each other. Notice that an \( A \)-path is non-returning if and only if its reverse is non-returning. A family of fully node-disjoint non-returning \( A \)-paths will be called a non-returning family. Let \( \nu(G, A, \omega, \pi) \) denote the maximum cardinality of a non-returning family.

The problem of finding a maximum non-returning family is a slight generalization of the problem of packing non-zero \( A \)-paths in group-labeled graphs. To see this, we define the set of potentials as the set of elements of the group, the potential of origin as zero for each terminal, and we define \( \pi(e) \) as the multiplication by the group-label on arc \( e \).

2. The min-max formula

Consider a graph \( G = (V, E) \) with a set \( A \subseteq V \) of terminals. Let \( \tilde{\nu}(G, A) \) denote the maximum number of fully node-disjoint \( A \)-paths. This is in fact a special case of packing non-returning \( A \)-paths, since it is easy to construct mappings of potentials such that every \( A \)-path is non-returning. T. Gallai [2] determined \( \tilde{\nu}(G, A) \) by a reduction to non-bipartite matching.

Next we define a notion which we use in the main theorem to determine \( \nu(G, A, \omega, \pi) \). Consider a set \( F \subseteq E \) of arcs. Let \( A' := A \cup V(F) \). \( F \) is called \( A \)-balanced if \( \omega \) can be extended to a function \( \omega' : A' \to \Omega \) such that \( \pi(ab)(\omega'(a)) = \omega'(b) \) for each arc \( ab = e \in F \). (Or equivalently, each arc in \( F \) gives a one-arc returning \( A' \)-path with respect to \( \omega' \). Notice that an \( A \)-path \( P \) is returning if and only if \( E(P) \) is \( A \)-balanced.)

**Theorem 2.1.** If \( G, A, \omega, \pi \) is given as above then the equation

\[
\nu(G, A, \omega, \pi) = \min \tilde{\nu}(G - F, A \cup V(F))
\]

holds, where the minimum is taken over \( A \)-balanced arc-sets \( F \).