OPTIMAL STRONG PARITY EDGE-COLORING
OF COMPLETE GRAPHS

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A parity walk in an edge-coloring of a graph is a walk along which each color is used an even number of times. Let \( p(G) \) be the least number of colors in an edge-coloring of \( G \) having no parity path (a parity edge-coloring). Let \( \tilde{p}(G) \) be the least number of colors in an edge-coloring of \( G \) having no open parity walk (a strong parity edge-coloring). Always \( \tilde{p}(G) \geq p(G) \geq \chi'(G) \). We prove that \( \tilde{p}(K_n) = 2^{\lceil \log_2 n \rceil} - 1 \) for all \( n \). The optimal strong parity edge-coloring of \( K_n \) is unique when \( n \) is a power of 2, and the optimal colorings are completely described for all \( n \).

1. Introduction

Our work began by studying which graphs embed in the hypercube \( Q_k \), the graph with vertex set \( \{0, 1\}^k \) in which vertices are adjacent when they differ in exactly one coordinate. Coloring each edge with the position of the bit in which its endpoints differ imposes two necessary conditions for the coloring inherited by a subgraph \( G \):

1) every cycle uses each color an even number of times,
2) every nontrivial path uses some color an odd number of times.

Existence of a \( k \)-edge-coloring satisfying conditions (1) and (2) is also sufficient for a connected graph \( G \) to be a subgraph of \( Q_k \). This characterization

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of subgraphs of $Q_k$ appeared in 1972 (Havel and Morávek [8]). The problem was studied as early as 1953 (Shapiro [13]).

Let the usage of a color on a walk be the parity of the number of times it appears along the walk. A parity walk is a walk in which the usage of every color is even. Condition (1) above states that every cycle is a parity walk, and (2) states that no path is a parity walk.

In general, a parity edge-coloring is an edge-coloring with no parity path, and a strong parity edge-coloring (spec) is an edge-coloring with no open parity walk (that is, every parity walk is closed). Some graphs embed in no hypercube, but giving the edges distinct colors produces a spec for any graph. Hence the parity edge-chromatic number $p(G)$ and the strong parity edge-chromatic number $\hat{p}(G)$, defined respectively to be the minimum numbers of colors in a parity edge-coloring of $G$ and in a spec of $G$, are well defined. Elementary results on these parameters appear in [5].

When $T$ is a tree, $\hat{p}(T) = p(T) = k$, where $k$ is the least integer such that $T$ embeds in $Q_k$ [5]. Since incident edges of the same color would form a parity path of length 2, every parity edge-coloring is a proper edge-coloring, and hence $p(G) \geq \chi'(G)$, where $\chi'(G)$ denotes the edge-chromatic number. Although there are graphs $G$ with $\hat{p}(G) > p(G)$ [5], it remains unknown how large $\hat{p}(G)$ can be when $p(G) = k$. It also remains unknown whether there is a bipartite graph $G$ with $\hat{p}(G) > p(G)$.

When $n$ is a power of 2, we will prove that the complete graph $K_n$ has a unique optimal spec (up to isomorphism), which will help us determine $\hat{p}(K_n)$ for all $n$. With a suitable naming of the vertices, we call this edge-coloring of $K_n$ the “canonical” coloring.

**Definition 1.1.** For $A \subseteq \mathbb{F}_2^k$, let $K(A)$ be the complete graph with vertex set $A$. The canonical coloring of $K(A)$ is the edge-coloring $f$ defined by $f(uv) = u + v$, where $u + v$ is binary vector addition. When $n = 2^k$, letting $A = \mathbb{F}_2^k$ yields the canonical coloring of $K_n$.

**Lemma 1.2.** For $A \subseteq \mathbb{F}_2^k$, the canonical coloring of $K(A)$ is a spec. Consequently, if $n = 2^k$, then $\hat{p}(K_n) = p(K_n) = \chi'(K_n) = n - 1$.

**Proof.** If $W$ is an open walk, then its endpoints differ in some bit $i$. Thus in the canonical coloring the total usage of colors flipping bit $i$ along $W$ is odd, and hence some color has odd usage on $W$. The canonical coloring of $K(\mathbb{F}_2^k)$ uses $2^k - 1$ colors (the color $0^k$ is not used). The lower bound follows from $\hat{p}(G) \geq p(G) \geq \chi'(G) \geq \Delta(G)$.

Since every complete graph is a subgraph of the next larger complete graph, we obtain $\hat{p}(K_n) \leq 2^{\lceil \log n \rceil} - 1$. In Section 2, we prove that this upper