THE THREE-IN-A-TREE PROBLEM

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We show that there is a polynomial time algorithm that, given three vertices of a graph, tests whether there is an induced subgraph that is a tree, containing the three vertices. (Indeed, there is an explicit construction of the cases when there is no such tree.) As a consequence, we show that there is a polynomial time algorithm to test whether a graph contains a “theta” as an induced subgraph (this was an open question of interest) and an alternative way to test whether a graph contains a “pyramid” (a fundamental step in checking whether a graph is perfect).

1. Introduction

All graphs in this paper are finite and simple. If \( G \) is a graph, its vertex- and edge-sets are denoted \( V(G), E(G) \). If \( X \subseteq V(G) \), the subgraph with vertex set \( X \) and edge set all edges of \( G \) with both ends in \( X \) is denoted \( G|X \), and called the subgraph induced on \( X \).

There are many algorithmic questions of interest concerning the existence of an induced subgraph of some specific type containing some specified vertices, but almost all of them seem to be NP-complete, by virtue of the following result of Bienstock [1]:

1.1. The following problem is NP-complete:

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Input: A graph $G$ and two edges $e, f$ of $G$.

Question: Is there a subset $X \subseteq V(G)$ such that $G|X$ is a cycle containing $e, f$?

Bienstock’s result leaves very little room between the trivial problems and the NP-complete problems, but in this paper we report on a problem that falls into the gap. We call the following the “three-in-a-tree” problem:

Input: A graph $G$ and three vertices $v_1, v_2, v_3$ of $G$.

Question: Does there exist $X \subseteq V(G)$ with $v_1, v_2, v_3 \in X$ such that $G|X$ is a tree?

For most graphs one would expect a “yes” answer, but there are interesting graphs for which the answer is “no”; for instance, if $e_1, e_2, e_3$ are edges of a graph $H$ each incident with a vertex of degree one, and $G$ is the line graph of $H$, then $e_1, e_2, e_3$ are vertices of $G$ and there is no induced tree in $G$ containing $e_1, e_2, e_3$. Nevertheless, we will show that the three-in-a-tree problem can be solved in time $O(|V(G)|^4)$. We shall give an explicit construction of all instances $(G, v_1, v_2, v_3)$ such that the desired tree does not exist, and the proof that all such instances must fall under this construction can be converted to an algorithm to check whether the desired tree exists or not.

2. Thetas, pyramids and prisms

We were led to the three-in-a-tree problem while working on the question of deciding if a graph contains a theta, so let us describe that. First we need some definitions. If $G, H$ are graphs, and $H$ is isomorphic to $G|X$ for some $X \subseteq V(G)$, we say that $G$ contains $H$ as an induced subgraph. A path is a graph $P$ whose vertex set and edge set can be labeled as $V(P) = \{v_1, \ldots, v_k\}$ and $E(P) = \{e_1, \ldots, e_{k-1}\}$ for some $k \geq 1$, such that $e_i$ is incident with $v_i, v_{i+1}$ for $1 \leq i \leq k-1$. A cycle is a graph $C$ with $V(C) = \{v_1, \ldots, v_k\}$ and $E(C) = \{e_1, \ldots, e_k\}$ for some $k \geq 3$, such that $e_i$ is incident with $v_i, v_{i+1}$ for $1 \leq i \leq k-1$, and $e_k$ is incident with $v_1, v_k$. The length of a path or cycle is the number of edges in it, and a path or cycle is odd or even if its length is odd or even respectively. A path or cycle of $G$ means a subgraph (not necessarily induced) of $G$ that is a path or cycle. A hole of $G$ means a cycle in $G$ that is an induced subgraph and has length at least four. A triangle is a set of three pairwise adjacent vertices.

Here are three types of graph that will be important to us: