B-Valued fuzzy variable

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Abstract In this paper, the concept of B-valued fuzzy variable is first presented. Then, some mathematical properties of B-valued fuzzy variable are also investigated, including independence, identical distribution, expected value, variance, inequalities, convergence concepts, and so on.

Keywords B-Valued fuzzy variable · Credibility measure · Expected value · Variance

1 Introduction

There are two methods to describe fuzzy phenomena in mathematical ways, one is “fuzzy set”, the other is “fuzzy variable”. The term of fuzzy set was presented by Zadeh (1965), which is defined via membership function. Additionally, to describe the chance of a fuzzy event, possibility measure was presented by Zadeh (1978) in 1978. In order to develop fuzzy set theory on the basis of axiomatic foundation, Nahmias (1978) presented three axioms to define possibility measure and also gave the concept of possibility space. Fuzzy variable is defined as a function from the possibility space to the set of real numbers to describe fuzzy phenomenon. Hence, fuzzy set theory can be developed based on the axiomatic foundation, just as the development of probability theory (refer to Laha and Rohatgi 1979; Loève 1977). Actually, fuzzy set and fuzzy variable coincide with each other. For any fuzzy set, there exists a fuzzy variable whose membership function is just that of fuzzy set. Conversely, for any fuzzy variable, we can deduce its membership function by using possibility measure.

Although possibility measure has been widely used in theory and applications, it is not a self-dual measure. However, self-duality property is absolutely necessary for the further development of fuzzy mathematics. For this reason, Liu and Liu (2002) presented the concept of credibility measure, which is defined as the average of possibility measure and necessity measure. Additionally, a sufficient and necessary condition for credibility measure was given by Li and Liu (2006). On the basis of credibility measure, credibility theory was founded by Liu (2004) as a branch of mathematics to study the behavior of fuzziness. For more details of theory and applications of credibility theory, we may refer to Gao (2007), Liu (2006, 2007), Liu and Zhu (2007), Yang et al. (2007); Yang and Liu (2007), and so on.

Up to now, credibility theory has been studied on the real line. All the well-known consequences, for instance, expected value, variance, inequalities, independence and convergence, concern real-valued fuzzy variables. Some of these results can also be extended to finite-dimensional fuzzy variable without much difficulty. But for the general linear space, some problems may arise. In this paper, we shall investigate some properties of fuzzy variable taking values in normed linear space, i.e., B-valued fuzzy variable.

The rest of this paper is organized as follows. After defining the concept of B-valued fuzzy variable in Sect. 2, we investigate independence and identical distribution of B-valued fuzzy variables in Sect. 3. Section 4 explores expected value operator and variance for B-valued fuzzy variable. Section 5 investigates some inequalities with respect to expected value and variance. At last, several convergence concepts for the sequence of B-valued fuzzy variables are introduced.
2 Credibility space and $\mathcal{B}$-valued fuzzy variable

Let $\Theta$ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of $\Theta$, and $\text{Pos}$ the possibility measure on $\mathcal{P}(\Theta)$. The necessity measure of set $B$ is defined as the impossibility of the opposite set $B^c$, i.e.,

$$\text{Nec}\{B\} = 1 - \text{Pos}\{B^c\}, \quad B \in \mathcal{P}(\Theta).$$

The credibility measure $Cr$ on $\mathcal{P}(\Theta)$, defined by Liu and Liu (2002), is the average of possibility measure and necessity measure, i.e.,

$$Cr\{B\} = \frac{1}{2}\left(\text{Pos}\{B\} + \text{Nec}\{B\}\right), \quad B \in \mathcal{P}(\Theta).$$

In 2006, Li and Liu (2006) investigated a sufficient and necessary condition for credibility measure as follows.

**Theorem 2.1** (Li and Liu 2006) A set function $Cr$ on $\mathcal{P}(\Theta)$ is a credibility measure if and only if it satisfies the following conditions:

(i) $Cr\{\emptyset\} = 1$;
(ii) $Cr\{A\} \leq Cr\{B\}$ whenever $A \subset B$;
(iii) $Cr\{B\} + Cr\{B^c\} = 1$ for any $B \in \mathcal{P}(\Theta)$;
(iv) $Cr\{\bigcup_{i=1}^{n} B_i\} = \sup_{\theta} \left(\text{Cr}\{B_i\}\right)$ for any $\{B_i\} \subset \mathcal{P}(\Theta)$ with $\sup_{\theta} \text{Cr}\{B_i\} \leq 0.5$.

**Definition 2.1** (Liu 2004) Let $\Theta$ be a nonempty set, $\mathcal{P}(\Theta)$ the power set of $\Theta$, and $Cr$ a credibility measure. Then the triplet $(\Theta, \mathcal{P}(\Theta), Cr)$ is called a credibility space, and a set $B$ with $B \in \mathcal{P}(\Theta)$ is called a fuzzy event.

**Definition 2.2** (Liu 2004) A real-valued fuzzy variable $\xi$ is defined as a function from a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ to the set of real numbers.

Let $\mathbb{N}$ represent the set of real numbers. For any subset $B$ of $\mathbb{N}$, the set

$$\{\theta \in \Theta | \xi(\theta) \in B\}$$

is a fuzzy event, which is denoted by “$\xi \in B$” for simplicity.

**Definition 2.3** (Liu and Liu 2002) Let $\xi$ be a real-valued fuzzy variable. Then the expected value of $\xi$ is defined by

$$E[\xi] = \int_{-\infty}^{+\infty} \text{Cr}\{\xi \geq r\} dr - \int_{-\infty}^{0} \text{Cr}\{\xi \leq r\} dr$$

provided that at least one of the two integrals is finite.

**Definition 2.4** (Liu and Gao 2007) The real-valued fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ are said to be independent if

$$\text{Cr}\left\{\bigcap_{i=1}^{n} [\xi_i \in B_i]\right\} = \min_{1 \leq i \leq n} \text{Cr}\{\xi_i \in B_i\}$$

for any sets $B_1, B_2, \ldots, B_n \subset \mathbb{N}$.

**Theorem 2.2** (Liu 2004) Let $\xi_1, \xi_2, \ldots, \xi_n$ be independent real-valued fuzzy variables, and $f_1, f_2, \ldots, f_n$ real-valued functions. Then $f_1(\xi_1), f_2(\xi_2), \ldots, f_n(\xi_n)$ are independent real-valued fuzzy variables.

**Theorem 2.3** (Liu and Liu 2002) Let $\xi$ and $\eta$ be independent real-valued fuzzy variables with finite expected values. Then for any real numbers $a$ and $b$, we have

$$E[a\xi + b\eta] = aE[\xi] + bE[\eta].$$

We can extend the real-valued fuzzy variable to the real Banach space. That is, fuzzy variable assumes values in the real Banach space, which is called $\mathcal{B}$-valued fuzzy variable. In the following, let $\mathcal{B}$ denote the real Banach space with the norm $\|\cdot\|$, $\mathcal{B}^*$ the dual of $\mathcal{B}$ (That is, $\mathcal{B}^*$ is the Banach space consisting of all bounded (continuous) linear functionals on $\mathcal{B}$).

**Definition 2.5** A $\mathcal{B}$-valued fuzzy variable $\xi$ is a function from a credibility space $(\Theta, \mathcal{P}(\Theta), Cr)$ to the real Banach space $\mathcal{B}$.

**Theorem 3.1** Let $\xi$ be a $\mathcal{B}$-valued fuzzy variable. Then $\|\xi\|$ is a real-valued fuzzy variable.

**Proof** For each $\theta \in \Theta$, we have $\xi(\theta) \in \mathcal{B}$. Then $\|\xi(\theta)\| \in \mathbb{R}$. Thus $\|\xi\|$ is a function from the credibility space to the set of real numbers, which implies $\|\xi\|$ is a real-valued fuzzy variable. The proof is thus completed.

**Theorem 3.2** Let $\xi$ be a $\mathcal{B}$-valued fuzzy variable. Then for any $l \in \mathcal{B}^*$, $l(\xi)$ is a real-valued fuzzy variable.

**Proof** For each $\theta \in \Theta$, we have $l(\xi) \in \mathcal{B}$. Since $l$ is a bounded (continuous) linear functional on $\mathcal{B}$, we have $l(\xi(\theta)) \in \mathbb{R}$. Thus $l(\xi)$ is a function from the credibility space to the set of real numbers, which implies $l(\xi)$ is a real-valued fuzzy variable. The proof is thus completed.

3 Independence and identical distribution

**Definition 3.1** A finite set of $\mathcal{B}$-valued fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ is said to be independent if for any sets $B_1, B_2, \ldots, B_n \subset \mathcal{B}$, we have

$$\text{Cr}\left\{\bigcap_{i=1}^{n} [\xi_i \in B_i]\right\} = \bigwedge_{i=1}^{n} \text{Cr}\{\xi_i \in B_i\}.$$

More generally, a collection of $\mathcal{B}$-valued fuzzy variables is said to be independent if every finite subset is independent.

**Theorem 3.3** $\mathcal{B}$-valued fuzzy variables $\xi_1, \xi_2, \ldots, \xi_n$ are independent if and only if

$$\text{Cr}\left\{\bigcup_{i=1}^{n} [\xi_i \in B_i]\right\} = \bigvee_{i=1}^{n} \text{Cr}\{\xi_i \in B_i\}$$

for any sets $B_1, B_2, \ldots, B_n \subset \mathcal{B}$.