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Palais-Smale sequence relative to the Trudinger-Moser inequality

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Abstract. We study the Palais-Smale sequence relative to the two-dimensional Trudinger-Moser inequality. First, we show that such a sequence concentrates to finite points if it is not compact in $H^1$. Then, under an additional condition, the singular limit is specified in use of the Green’s function.

1 Introduction

Several questions of physics, geometry, and biology are formulated in terms of elliptic equations with the exponential nonlinearity. A typical example is

$$-\Delta v = \frac{\lambda K(x)e^v}{\int_{\Omega} K(x)e^v \, dx} \quad \text{in} \quad \Omega \quad \text{with} \quad v = 0 \quad \text{on} \quad \partial \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, and $K(x) > 0$ is a smooth function defined on $\Omega$. It is related to the complex function theory and the theory of surfaces ([2], [32]), but if $K(x) \equiv 1$ it is the mean field equation of vortex points in Onsager’s formulation of statistical mechanics ([16], [5], [6]).

Another example is the prescribed Gaussian curvature equation studied by Kazdan and Warner [14],

$$-\Delta_g v = \lambda \left( \frac{V e^v}{\int_{\mathcal{M}} V e^v \, dv_g} - W \right) \quad \text{on} \quad \mathcal{M},$$

where $(\mathcal{M}, g)$ denotes a compact Riemannian surface, and $V$ and $W$ are smooth functions on $\mathcal{M}$ satisfying $V > 0$ somewhere and $\int_{\mathcal{M}} W \, dv_g = 1$. If $(\mathcal{M}, g)$ is a flat torus, $\lambda = 4\pi$, $W = 1/|\mathcal{M}|$, and $V = \exp u_0$ with $u_0 = u_0(x)$ satisfying

$$-\Delta u_0 = \frac{4\pi N}{|\mathcal{M}|} - 4\pi \sum_{j=1}^N \delta_{p_j}(dx) \quad \text{on} \quad \mathcal{M} \quad \text{with} \quad \int_{\mathcal{M}} u_0 = 0$$

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for $N \in \mathcal{N}$, then it is the problem introduced by Tarantello [33] as a limiting state in Chern-Simons-Higgs’ gauge theory concerning the super conductivity in high temperature. Nirenberg’s problem is formulated similarly, of which details we refer to [21], [7], [1].

We also have the problem

$$-\Delta v + av = \frac{\lambda K(x)e^v}{\int_{\Omega} K(x)e^v dx} \quad \text{in } \Omega \quad \text{with} \quad \frac{\partial v}{\partial \nu} = 0 \quad \text{in } \partial \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a bounded domain with smooth boundary $\partial \Omega$, $a > 0$ is a positive constant, $K(x) > 0$ is a smooth function defined on $\Omega$, and $\nu$ is the outer unit normal vector on $\partial \Omega$. It was introduced by Childress and Percus [11] as the equilibrium state of the Keller-Segel system [15]. That system was proposed to describe aggregation of slime molds sensitive to some chemical substance secreted by themselves, where $K(x)$ acts as an environment function.

Those problems have variational structures of their own. For instance, $v = v(x)$ is a solution to (1) with $K(x) \equiv 1$ if and only if it is a critical point of the functional

$$\tilde{J}_\lambda(v) = \frac{1}{2} \|\nabla v\|^2_2 - \lambda \log \left( \int_{\Omega} e^v dx \right)$$

defined for $v \in H^1_0(\Omega)$. Here and henceforth, $\|\cdot\|_p$ is the standard $L^p(\Omega)$ norm and $\int_{\Omega} = \frac{1}{|\Omega|} \int_{\Omega}$. Then, Trudinger-Moser’s inequality [20] assures its global minimizer for $\lambda \in (0, 8\pi)$. We have a solution to (1) in this case, but recently, Ding, Jost, Li, and Wang [12] showed that this problem has a solution even for $\lambda \in (8\pi, 16\pi)$ if $\Omega$ has a hole. Non-constant solutions to (2) on flat torus with $u_0 \equiv 0$ and to (3) on bounded domain $\Omega$ with smooth boundary $\partial \Omega$ are obtained by Struwe and Tarantello [30] and Senba and Suzuki [26], respectively.

Thus, those works are devoted to finding non-trivial solutions and employ similar arguments. That is, first, existence of the solution is assured for a.e. $\lambda > 0$ by Struwe’s argument in the mini-max setting of variation. Then, the solution is obtained for the “non-quantized value” of $\lambda$ in use of the blowup analysis to the family of solutions. The arguments of Jeanjean and Tolland [13] for the former and of Brezis and Merle [3], Li and Shafrir [18] for the latter, are available, respectively.

Those cases are studied also by Nagasaki and Suzuki [22], Ye [34], Ma and Wei [19], Li [17], Chen and Lin [9]. Roughly speaking, those works assert that extra constraints to the family such as boundary conditions make blowup points to be simple and that in some cases their locations are controlled by the Green’s functions. A counter part was given by Chen [10], where the multiple blowup is shown to occur in the general case. Here, we get the question to characterize sequences provided with such properties. If any non-compact Palais-Smale sequence has the quantized mass $\lambda$, then the proof of the existence of the solution will become simpler.

In this paper we show that the Palais-Smale sequence has those properties if an additional condition is imposed. Unfortunately, that condition looks optimal, but so far we can refine previous works to some extents.