A variational characterization for $\sigma_{n/2}$

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Abstract. We present here a conformal variational characterization in dimension $n = 2k$ of the equation $\sigma_k(A_g) = \text{constant}$, where $A$ is the Schouten tensor. Using the fully nonlinear parabolic flow introduced in [3], we apply this characterization to the global minimization of the functional.

1 Introduction

Let $(M^n, g)$ be a compact, connected, locally conformally flat Riemannian manifold of even dimension $n$, and let the Ricci tensor and scalar curvature be denoted by $\text{Ric}$ and $R$, respectively. We recall the definition of the Schouten tensor

$$A_g = \frac{1}{n-2} \left( \text{Ric} - \frac{1}{2(n-1)} R g \right),$$

and we consider the equation

$$\sigma_k(A_{\tilde{g}}) = \text{constant},$$

where $\tilde{g} = e^{2u} g$ is a conformal metric. A variational characterization for this equation was given in [4] for $k \neq n/2$, and this problem was further studied by P. Guan and G. Wang in [3], where they proposed the following conformal flow:

$$\frac{d}{dt} g = - (\log \sigma_k(A_g) - \log r_k) \cdot g,$$

$$g(0) = g_0,$$

where

$$\log r_k = \frac{1}{Vol(g)} \int_M \log \sigma_k(g) \, dvol(g).$$

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This is fully nonlinear parabolic equation as long as the Schouten tensor of \( g \) belongs to the set

\[
\Gamma_k^+ = \text{component of } \{ \sigma_k > 0 \} \text{ containing the positive cone.}
\]

P. Guan and G. Wang established global existence of the flow, and for \( k \neq n/2 \) they proved convergence in \( C^\infty \) to a solution of \( \sigma_k = \text{constant} \). In this note, we extend their result to the remaining case \( k = n/2 \).

**Theorem.** For every locally conformally flat initial metric \( g_0 \), with \( A_{g_0} \in \Gamma_k^+ \), the evolution equation (2) has a global solution which converges in \( C^\infty \) to a limiting metric \( g \) satisfying \( \sigma_{n/2}(A_g) = \text{constant} \).

The missing element in Guan and Wang’s argument is the conformal primitive for \( \sigma_{n/2}(A_g) \). The purpose of this paper is to present a derivation of such a functional. A different method was used by S.-Y. A. Chang and P. Yang in [1].

### 2 Variational characterization

In the first step, we prove existence of a conformal primitive by verifying the integrability conditions. Let \( \mathcal{M} \) be an equivalence class of conformally equivalent metrics on \( M \). The tangent space of \( \mathcal{M} \) at \( g \) can be identified with the space of real-valued functions on \( M \). We define a 1-form \( \alpha \) on \( \mathcal{M} \) by

\[
\alpha_g(v) = \int_M \sigma_{n/2}(A_g) v \, dvol_g,
\]

where \( A_g \) denotes the Schouten tensor of the metric \( g \). We claim that \( \alpha \) is closed. To check this, we consider a two-parameter family of functions \( u(s,t) \) satisfying \( u(0,0) = 0 \). We define a family of conformal metrics \( \tilde{g}(s,t) \) by

\[
\tilde{g}(s,t) = e^{2u(s,t)} g.
\]

The exterior derivative of \( \alpha \) is given by

\[
d\alpha \left( \frac{\partial}{\partial s}, \frac{\partial}{\partial t} \right) = \frac{d}{ds} \alpha \left( \frac{\partial}{\partial t} \right) - \frac{d}{dt} \alpha \left( \frac{\partial}{\partial s} \right).
\]

Using the transformation formulae

\[
A_{\tilde{g}} = -\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g + A_g,
\]

and

\[
\tilde{\nabla}^2 h = \nabla^2 h - du \otimes dh - dh \otimes du + \langle du, dh \rangle g,
\]

(3) and

(4)