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Uniqueness and bubbling of the 2-dimensional Landau-Lifshitz flow

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Abstract. We consider the Landau-Lifshitz flow on a bounded planar domain. An \( \varepsilon \)-regularity type a-priori estimate provides the analytic tool for the subsequent geometric description of the flow at isolated singularities. At forward isolated singularities where the energy is not left continuous the flow concentrates energy and develops bubbles. As in J.Qing’s bubbling-energy-equality for the harmonic map flow, the energy loss at such a singularity can be recovered as a finite sum of energies of tangent bubbles. We then clarify a known uniqueness result for the Landau-Lifshitz flow and show how non-uniqueness of extensions of the flow after point singularities is related to backward bubbling. Finally the \( \varepsilon \)-regularity estimate also yields a partial compactness result for sequences of smooth solutions to the Landau-Lifshitz flow with uniformly bounded energy, defined on a planar domain.

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1. Introduction

The Landau-Lifshitz flow \( u : \overline{\Omega} \times \mathbb{R}_+ \to S^2 \) is defined by

\[
\begin{align*}
\partial_t u &= -\alpha u \times (u \times \Delta u) + \beta u \times \Delta u \quad \text{in } \Omega \times \mathbb{R}_+, \\
u &= u_0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial \Omega \times \mathbb{R}_+),
\end{align*}
\]

where \( \alpha > 0, \beta \in \mathbb{R} \) and where “\( \times \)” denotes the usual vector product in \( \mathbb{R}^3 \). Here \( \Omega \subset \mathbb{R}^2 \) denotes a smooth bounded domain and \( S^2 \subset \mathbb{R}^3 \) is the standard sphere.

In physics these equations describe an isotropic Heisenberg spin chain phenomenon in non-equilibrium magnetism (see [22]). The map \( u \) describes the spin density, \( \alpha > 0 \) is the Gilbert damping constant and \( \beta \in \mathbb{R} \) is an exchange constant.

By using that \( |u| \equiv 1 \), from the identity \( a \times (b \times c) = (a \cdot c) b - (a \cdot b) c \) for \( a, b, c \in \mathbb{R}^3 \) it is easy to see that for sufficiently regular solutions equation (1) is equivalent to

\[
\begin{align*}
\gamma_1 \partial_t u - \gamma_2 u \times \partial_t u &= \Delta u + |\nabla u|^2 u \quad \text{in } \Omega \times \mathbb{R}_+,
\end{align*}
\]

where \( \gamma_1 := \frac{\alpha}{\alpha^2 + \beta^2} > 0 \) and \( \gamma_2 := \frac{\beta}{\alpha^2 + \beta^2} \in \mathbb{R} \). (See [17].) By scaling the time variable, we may assume \( \gamma_1 = 1 \) and \( \gamma := \gamma_2 \in \mathbb{R} \).
For $\beta = 0$, or equivalently $\gamma = 0$, equation (3) is a harmonic map flow into $S^2$ with time scaled by $\alpha > 0$. (Compare [17].)

All our results also hold for the harmonic map flow with a general Riemannian manifold $N$ as target. This flow is given by

$$
\begin{align*}
\frac{\partial t}{u} - \Delta u &= A(u)(\nabla u, \nabla u) \quad \text{in } \Omega \times \mathbb{R}_+ \\
u &= u_0 \quad \text{on } (\Omega \times \{0\}) \cup (\partial \Omega \times \mathbb{R}_+),
\end{align*}
$$

where $A(u)(\nabla u, \nabla u) = \sum_{i=1}^2 A(u)(\partial_i u, \partial_i u)$ and $A$ denotes the second fundamental form of $N \hookrightarrow \mathbb{R}^n$. If $N = S^2 \hookrightarrow \mathbb{R}^3$ is the standard sphere, then $A(u)(\nabla u, \nabla u) = |\nabla u|^2 u$ and we recover (3) for $\gamma_1 = 1$ and $\gamma_2 = 0$.

Standard methods imply the existence of a smooth “short-time” solution

$$
u \in C^\infty(\Omega \times]0, T[; N)$$

to (2)-(3) for initial and boundary data $u_0 \in H^{1,2}(\Omega, N)$ and for sufficiently small $T = T(u_0, N) > 0$ (See [18, 31, 32, 17, 19]).

At the maximal existence time there are at most finitely many point singularities (compare Section 3, but also [31] and [18] for the harmonic map flow and [17] or [19] for the Landau-Lifshitz flow). By iterating the above short time existence result, a short time smooth solution can be extended to a global weak solution which is smooth except for finitely many point singularities and has decreasing energy

$$
t \mapsto E(u(t)) := \frac{1}{2} \int_{\Omega} |\nabla u|^2(x, t) \, dx.
$$

We will refer to this extension as the Struwe-solution. It was first constructed by M.Struwe in [31] for the harmonic map flow on Riemann surfaces. The construction was generalized to two dimensional domain manifolds with boundary by K.C.Chang in [2] and to the Landau-Lifshitz flow by B.Guo and M.C.Hong in [17]. See also [19] for the case with boundary.

Like the harmonic map flow (see [31] and [28]), at isolated singularities, where the energy is not left continuous, the Landau-Lifshitz flow splits off “bubbles”, i.e. non-constant harmonic maps $\varphi : S^2 \to S^2$, which account for the loss of energy at the singular time.

The Struwe-solution is unique in the class of solutions that are smooth except for isolated point singularities and with decreasing energy. A.Freire showed in [15] and [16] that it is still unique in the class

$$
H^{1,2}_{loc}(\Omega \times \mathbb{R}_+; N) \cap L^\infty(\mathbb{R}_+; H^{1,2}(\Omega; N))
$$

with initial and boundary data

$$
u_0 \in H^{1,2}(\Omega; N) \cap H^{3/2,2}(\partial \Omega; N),
$$

if the energy (6) is decreasing.

This uniqueness result was extended by Y. Chen and by B. Guo and S. Ding in [10, 9, 14, 5] to the Landau-Lifshitz flow, but the essential assumption that the energy