Abstract We establish the existence of infinitely many weak solutions for the
the one-dimensional version of the well-known and widely used Perona-Malik
anisotropic diffusion equation model in image processing. We establish the ex-
istence result under the homogeneous Neumann condition with smooth non-
constant initial values. Our method is to convert the problem into a partial dif-
ferential inclusion problem.

Keywords Perona-Malik model · One-dimensional · Infinitely many solutions ·
Differential inclusion · Relaxation property

1 Introduction

In this paper we establish the existence of infinitely many weak solutions for the
one-dimensional Perona-Malik anisotropic diffusion equation under the homoge-
neous Neumann boundary condition:

\[
\begin{align*}
& u_t = \sigma(u_x)_x, \quad (t, x) \in (0, T) \times (0, l) := Q_T, \\
& u(0, x) = u_0(x), \quad 0 \leq x \leq l, \\
& u_x(t, 0) = u_x(t, l) = 0, \quad 0 \leq t \leq T,
\end{align*}
\]

where \( \sigma(s) = s/(1 + s^2) \).

This work can be viewed as an unexpected application of the variational tech-
niques originated by Ball and James [6, 7] for the study of material microstructure
to forward-backward diffusion equations. The main idea is to rephrase (1.1) into a
The Perona-Malik anisotropic diffusion equation [28] was introduced in 1990 as an edge enhancement model in image processing. The model has a great impact in the study of image enhancement and edge detection by using evolutionary partial differential equations. It has motivated many new models and methods (see e.g. [3, 8, 29, 32] and references therein).

The original Perona-Malik equation is a two dimensional forward-backward diffusion equation in the form

\[ u_t(t, x) = \text{div}(\rho(|Du(t, x)|)Du(t, x)), \quad \text{in } (0, T) \times \Omega \]

under the homogeneous Neumann boundary condition \( \frac{\partial u}{\partial \nu} = 0 \) on \( \partial \Omega \), where \( \Omega \subset \mathbb{R}^2 \) is a square. Perona and Malik proposed two models for \( \rho(s) \), that is

- either \( \rho(s) = \frac{1}{1 + \left( \frac{s}{k} \right)^2} \), or \( \rho(s) = \exp\left(-\left(\frac{s}{k}\right)^2\right), \quad (k > 0) \).

In both cases, the Perona-Malik equations are diffusion equations of non-coercive and of forward-backward type. Let \( k = 1 \) for simplicity and let \( \sigma(s) = s/(1 + s^2) \), we see that \( \sigma(s) \) reaches its maximum at \( s = 1 \).

The main mathematical results on Perona-Malik model is mostly for the one-dimensional version (1.1). So far, there were no existence results when the initial datum \( u_0 \) has large derivative \( |(u_0)_t| \). In [19] it was proved that for small \( |(u_0)_t| \), (1.1) has a global smooth solution, as one only needs to use the increasing part of \( \sigma(\cdot) \) and the maximum principle. Also in [19], it was reported that an attempt to use the vanishing viscosity argument did not seem to produce a solution. It was shown in [17] that for large initial data, there are no \( C^1 \) solutions. In [5], a one-dimensional steady-state model of Perona-Malik equation was studied by using regularization and \( \Gamma \)-convergence, showing the formation of staircase function as a possible limit. A connection between Perona-Malik model and the Mumford-Shah functional was established via \( \Gamma \)-convergence in [22]. In a famous work on