Schwarz operators of minimal surfaces spanning polygonal boundary curves

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Abstract This paper examines the Schwarz operator $A$ and its relatives $\hat{A}$, $\bar{A}$ and $\overline{A}$ that are assigned to a minimal surface $X$ which maps consecutive arcs of the boundary of its parameter domain onto the straight lines which are determined by pairs $P_j, P_{j+1}$ of two adjacent vertices of some simple closed polygon $\Gamma \subset \mathbb{R}^3$. In this case $X$ possesses singularities in those boundary points which are mapped onto the vertices of the polygon $\Gamma$. Nevertheless it is shown that $A$ and its closure $\bar{A}$ have essentially the same properties as the Schwarz operator assigned to a minimal surface which spans a smooth boundary contour. This result is used by the author to prove in [Jakob, Finiteness of the set of solutions of Plateau’s problem for polygonal boundary curves. I.H.P. Analyse Non-lineaire (in press)] the finiteness of the number of immersed stable minimal surfaces which span an extreme simple closed polygon $\Gamma$, and in [Jakob, Local boundedness of the set of solutions of Plateau’s problem for polygonal boundary curves (in press)] even the local boundedness of this number under sufficiently small perturbations of $\Gamma$.

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1 Introduction and main results

This paper is concerned with the Schwarz operator

$$A \equiv A^X := -\Delta + 2KE$$ (1)
for a minimal surface $X$ which maps consecutive arcs of the boundary of its parameter domain onto the straight lines that are determined by pairs $P_j, P_{j+1}$ of two adjacent vertices of an arbitrarily fixed simple closed polygon $\Gamma \subset \mathbb{R}^3$ with $N+3$ vertices. Such a surface is given by a continuous $H^{1,2}$-mapping $X: \hat{B} \rightarrow \mathbb{R}^3$ of the closure of the unit disc $B := \{w = (u,v) \in \mathbb{R}^2 \mid |w| < 1\}$ into $\mathbb{R}^3$ which is harmonic on $B$, satisfies
\begin{equation}
|X_u| = |X_v|, \quad \langle X_u, X_v \rangle = 0 \quad \text{on } B
\end{equation}
and meets the boundary conditions $X(e^{i\theta}) \in \Gamma_j$ for $\theta \in [\tau_j, \tau_{j+1}], \ j = 1, \ldots, N + 3$, where $\Gamma_j$ denotes the line $\{P_j + t(P_{j+1} - P_j) \mid t \in \mathbb{R}\}$ and where the $\tau_j$ are consecutive angles in $(0,2\pi]$. We denote by $\hat{\mathcal{M}}(\Gamma)$ the set of such surfaces. Furthermore $K$ in (1) is the Gauss curvature of $X$ and $E := |X_u|^2$. For minimal surfaces $X$ bounded by some smooth contour $\Gamma$ the behaviour of $A^X$ is well known. The aim of this paper is to show that $A^X$ respectively its closure $\overline{A^X}$ have essentially the same properties for minimal surfaces $X$ with those “overshooting”, piecewise linear boundary values, as explained above. The author is using this result in [7,8] for his proof of the boundedness of the surfaces $\overline{A^X}$.

Finally we consider the assigned quadratic form
\begin{equation}
J(\varphi) = J^X(\varphi) := \int_B |\nabla \varphi|^2 + 2KE \varphi^2 d\omega
\end{equation}
which is defined for any $\varphi \in \dot{H}^{1,2}(B)$ due to $KE \in L^p(B)$ for some $p > 1$. To study the spectra of $A$ and $\hat{A}$ we investigate $J$ on the function space
\begin{equation}
S\dot{H}^{1,2}(B) := \{\varphi \in \dot{H}^{1,2}(B) \mid \|\varphi\|_{L^2(B)} = 1\}.
\end{equation}
Similarly we denote by $S(H^{2,2}(B) \cap \dot{H}^{1,2}(B))$ and $S\overline{\text{Dom}}(A)$ the intersections of the “$L^2(B)$-sphere” with the respective function spaces. Then we shall prove

**Theorem 1**  
(i) The spectra of $A$ and $\hat{A}$ coincide. They are discrete and accumulate only at $\infty$; thus their eigenspaces are finite dimensional. Furthermore for their common smallest eigenvalue $\lambda_{\min} := \lambda_{\min}(A) = \lambda_{\min}(\hat{A})$ we have
\begin{equation}
\lambda_{\min} = \inf_{S\overline{\text{Dom}}(A)} J = \inf_{S\dot{H}^{1,2}(B)} J = \inf_{S(H^{2,2}(B) \cap \dot{H}^{1,2}(B))} J.
\end{equation}