Minimizers of non convex scalar functionals and viscosity solutions of Hamilton-Jacobi equations

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Abstract We consider a class of non convex scalar functionals of the form

\[ F(u) = \int_{\Omega} f(x, u, Du) \, dx, \]

under standard assumptions of regularity of the solutions of the associated relaxed problem and of local affinity of the bipolar \( f^{**} \) of \( f \) on the set \( \{ f^{**} < f \} \). We provide an existence theorem, which extends known results to lagrangians depending explicitly on the three variables, by the introduction of integro-extremal minimizers of the relaxed functional which solve the equation

\[ f^{**}(x, u, Du) - f(x, u, Du) = 0, \]

or the opposite one, almost everywhere and in viscosity sense.

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1 Introduction

We consider the problem of minimizing functionals of the form

\[ F(u) = \int_{\Omega} f(x, u(x), Du(x)) \, dx, \]

where \( \Omega \) is an open bounded subset of \( \mathbb{R}^n \), \( f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) is a continuous function and the competing maps \( u : \Omega \to \mathbb{R} \) belong to some Sobolev space and satisfy a Dirichlet boundary condition.

Our interest is devoted to functionals with non convex integrand \( f \) for which the direct method of the Calculus of Variations does not provide automatically a minimizer. The study of
such functionals is usually approached by the following scheme: introduce the lower convex envelope $f^{**}$ of $f$ with respect to the last variable and the corresponding relaxed functional

$$
\mathcal{F}(u) = \int_\Omega f^{**}(x, u(x), Du(x))dx,
$$

which, by the classical argument of the direct method, admits a nonempty set $S$ of minimizers. Then the problem of minimizing $\mathcal{F}$ reduces to find an element of $S$ solving the equation

$$
f^{**}(x, u(x), Du(x)) - f(x, u(x), Du(x)) = 0
$$

for almost every $x \in \Omega$.

In the last years several steps have been made in this direction and we mention the papers quoted below and the monograph [6] for general setting.

The hypotheses used in the literature to treat the problem are essentially two: local affinity of the envelope $f^{**}$ on the set in which it is strictly less than $f$ and regularity, i.e. continuity and differentiability almost everywhere, of the elements of $S$. In [4,5,10,11,13] is considered a functional depending only on the gradient while in [3] it is studied a special class of functionals with dependence on the variables $u$ and $Du$ of sum type. Nonhomogeneous functionals (corresponding to lagrangians with an explicit dependence on the variables $x$ and $Du$) are treated in [9,12] while [2] is devoted to integrands with a general joint dependence on $u$ and $Du$. Up to now, no results are available for functionals whose lagrangian exhibits a general dependence on the three variables and in this paper we devote ourselves to this case.

In this paper, maintaining the approach sketched above and the aforementioned assumptions, we show that the method contained in [12,13], devoted to functionals whose lagrangian $f$ does not depend explicitly on the variable $u$, may be extended to the general case. The leading idea is to introduce the so called integro-extremal minimizers of $\mathcal{F}$, that is to say elements of $S$ which maximize or minimize the integral on $\Omega$ in the set $S$.

By this way we are able to give a rather simple and concise proof of a theorem which extends the existence results contained in the quoted papers and, in addition, we show that $\overline{u}$ is a viscosity solution of Eq. (1), while $\underline{u}$ is a viscosity solution of the equation

$$
f(x, u(x), Du(x)) - f^{**}(x, u(x), Du(x)) = 0.
$$

In paper [13] are considered special functionals with a lagrangian $g$ depending only on the gradient $Du$ and the well posedness of integro extremal minimizers is also treated, as a consequence of the structure of the epigraph of $g^{**}$. Assuming that the set on which $g^{**}$ is affine has a unique connected component, the pointwise supremum of two minimizers turns out to be still a minimizers and this implies that an integro-maximal [minimal] minimizer is also pointwise maximal [minimal], following from this fact its uniqueness. Imposing in addition suitable growth conditions on $g$, extremality of points $(\xi, g^{**}(\xi))$ such that $g^{**}(\xi) = g(\xi)$ in the epigraph of $g^{**}$ and local Lipschitz regularity of the boundary $\partial\Omega$ of the domain, we obtain also continuous dependence in strong Sobolev topology of integro extremal minimizers with respect to boundary data, equipped with uniform topology on $\partial\Omega$. In the present general case, by the dependence of the lagrangian $f$ on the three variables $(x, u, Du)$, the arguments of paper [13] cannot be directly reproduced, since it is no more true, in general, that the pointwise supremum of two minimizers is still a minimizer; hence the problem of well posedness of integro-extremal (viscosity) minimizers remains open.